MATHEMATIK

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Model Uncertainty and Expert Opinions in Continuous-Time Financial Markets

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Motivation

One should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand.[1]

– Rabbi Aha (ca. 4th century)

[W]e find that of the various optimizing models in the literature, there is no single model that consistently delivers a Sharpe ratio [. . .] higher than that of the 1/N portfolio [. . .].

– DeMiguel, Garlappi, Uppal (2009)

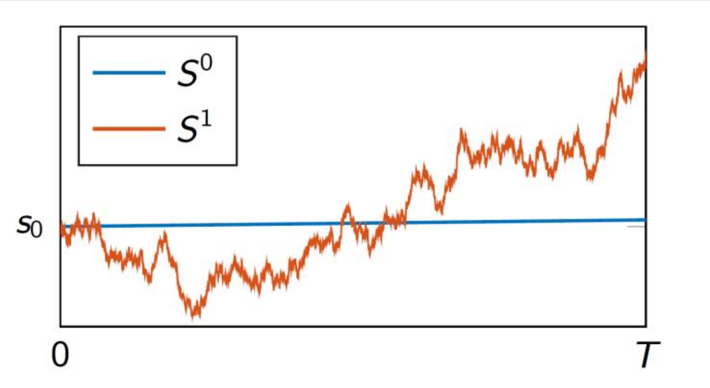
The 1/N investment strategy is optimal under high model ambiguity.

– Pflug, Pichler, Wozabal (2012)

[1] Babylonian Talmud: Tractate Baba Mezi'a, folio 42a.

The Merton Problem

Utility maximization in a Black–Scholes model



Utility maximization problem

$$V(x_0) = \sup_{\pi \in \mathcal{A}(x_0)} \mathrm{E}[U(X_T^\pi)]$$

- solved for most popular utility functions

Assumption that model parameters are known!

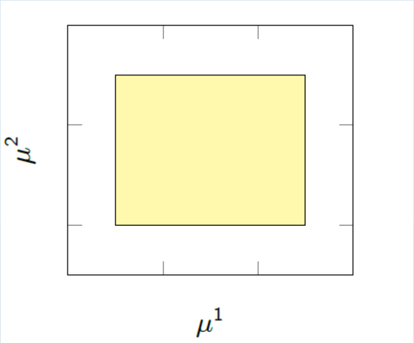
1. Model Uncertainty
2. Expert Opinions
3. Robust Optimization with Expert Opinions

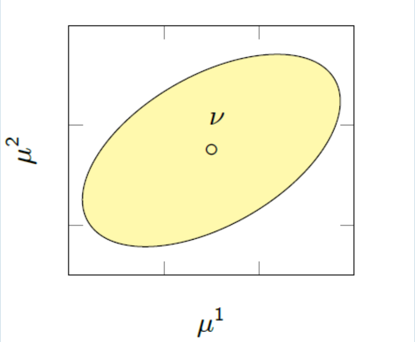
Financial Market Model

With drift uncertainty

Black–Scholes model

- one riskless asset, interest rate r
- $d \geq 2$ risky assets
$$\mathrm{d}S_t = \mathrm{diag}(S_t)\big(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mu}\big)$$
- wealth process $(X_t^{\pi})_{t\in[0,T]}$ corresponding to strategy π
- objective: maximize
$$\inf_{\mu\in K} \mathrm{E}_{\mu}[U(X_T^{\pi})]$$
over admissible strategies π

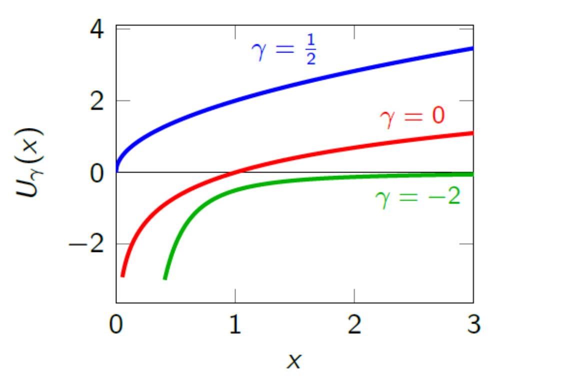




A First Result

It's best to not invest!

Let U_γ for $\gamma \in (-\infty, 1)$ denote power or logarithmic utility.



If $r\mathbf{1}_d \in K$, then $\pi \equiv 0$ is optimal for the worst-case optimization problem.

Constraint on admissible strategies:
$\langle \pi_t, \mathbf{1}_d \rangle = 1 \quad$ for all $t \in [0, T]$

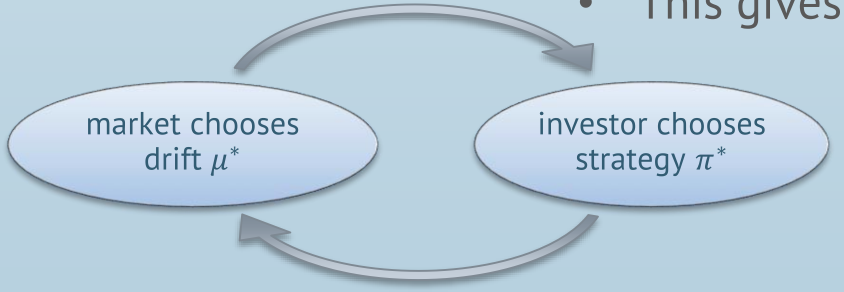
$$\sup_{\pi \in \mathcal{A}_1(x_0)} \inf_{\mu \in K} \mathrm{E}_\mu[U(X_T^\pi)]$$

How the Problem is Solved

A minimax theorem

- Find the best strategy given a fixed parameter $\mu \in K$
- Find the worst possible drift parameter $\mu^* \in K$
- This gives the solution to the dual problem

$$\inf_{\mu \in K} \sup_{\pi \in \mathcal{A}_1(x_0)} \mathrm{E}_\mu[U(X_T^\pi)]$$



$$\boxed{\sup_{\pi \in \mathcal{A}_1(x_0)} \inf_{\mu \in K} \mathrm{E}_\mu[U(X_T^\pi)] = \mathrm{E}_{\mu^*}[U(X_T^{\pi^*})] = \inf_{\mu \in K} \sup_{\pi \in \mathcal{A}_1(x_0)} \mathrm{E}_\mu[U(X_T^\pi)]}$$

Asymptotic Behavior for Large Uncertainty

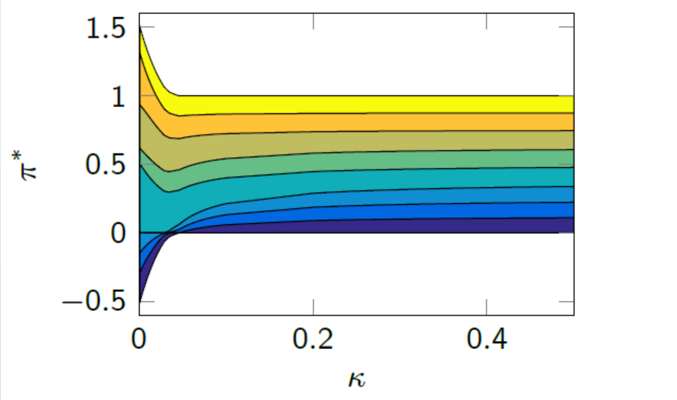
Convergence of the optimal strategy

For ellipsoid with radius κ:

$$\lim_{\kappa\to\infty} \pi_t^*(\kappa) = \frac{1}{\mathbf{1}_d^\top \Gamma^{-1}\mathbf{1}_d}\Gamma^{-1}\mathbf{1}_d$$

In particular for a ball:

$$\lim_{\kappa\to\infty} \pi_t^*(\kappa) = \frac{1}{d}\mathbf{1}_d$$



The optimal strategy converges to a generalized uniform diversification strategy as the level of uncertainty goes to infinity.

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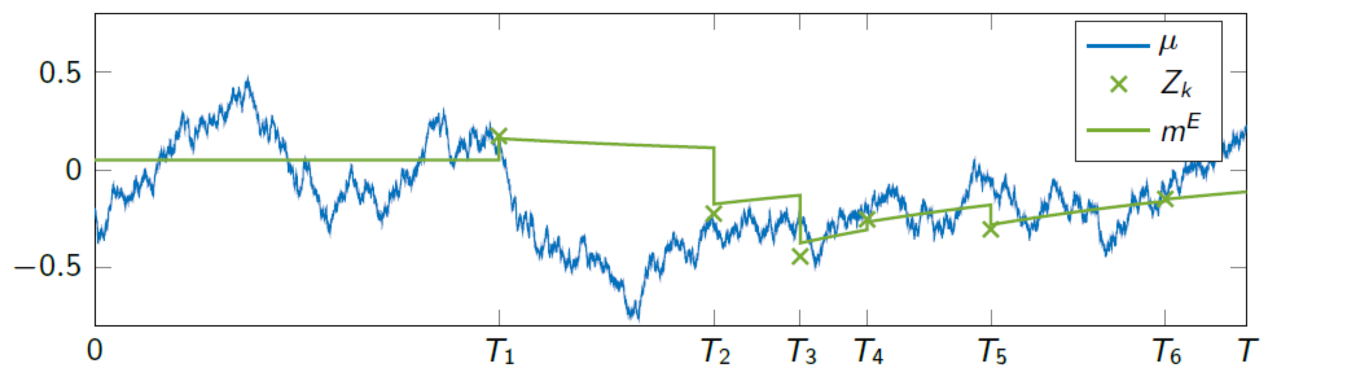
Financial Market Model

Stochastic drift and expert opinions

Ornstein–Uhlenbeck drift process:

$$dS_t = \operatorname{diag}(S_t)(\mu_t\,dt + \sigma_R\,dW_t^R)$$
$$d\mu_t = \alpha(\delta - \mu_t)\,dt + \beta\,dB_t$$



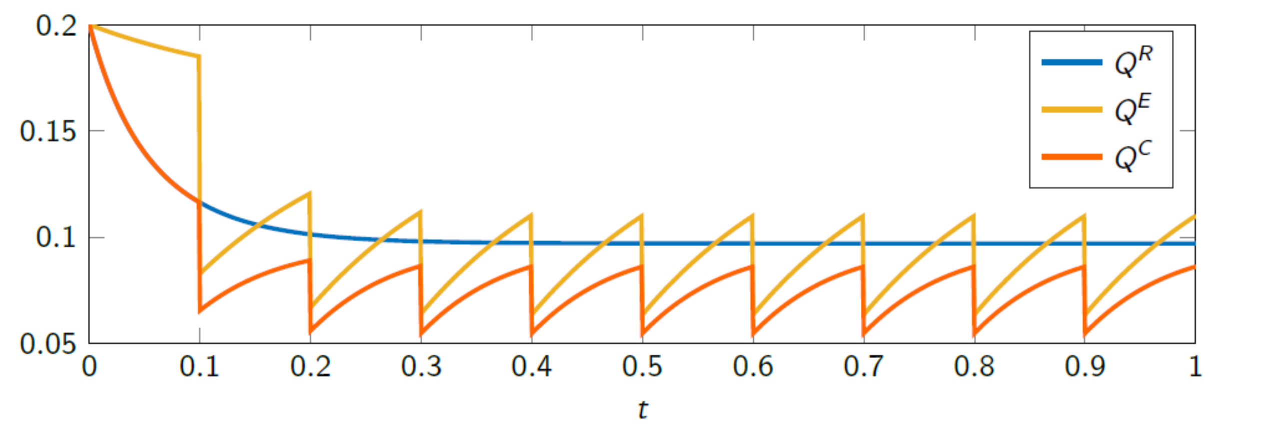
At discrete time points T_k: expert opinions $Z_k \mid \mu_{T_k} \sim \mathcal{N}(\mu_{T_k}, \Gamma_k)$

Filtering

Conditional mean and conditional covariance matrix

For investor filtration $(\mathcal{F}_t^H)_{t\geq 0}$ define

- $m_t^H = \mathrm{E}[\mu_t \mid \mathcal{F}_t^H]$
- $Q_t^H = \mathrm{E}[(\mu_t - m_t^H)(\mu_t - m_t^H)^\top \mid \mathcal{F}_t^H]$



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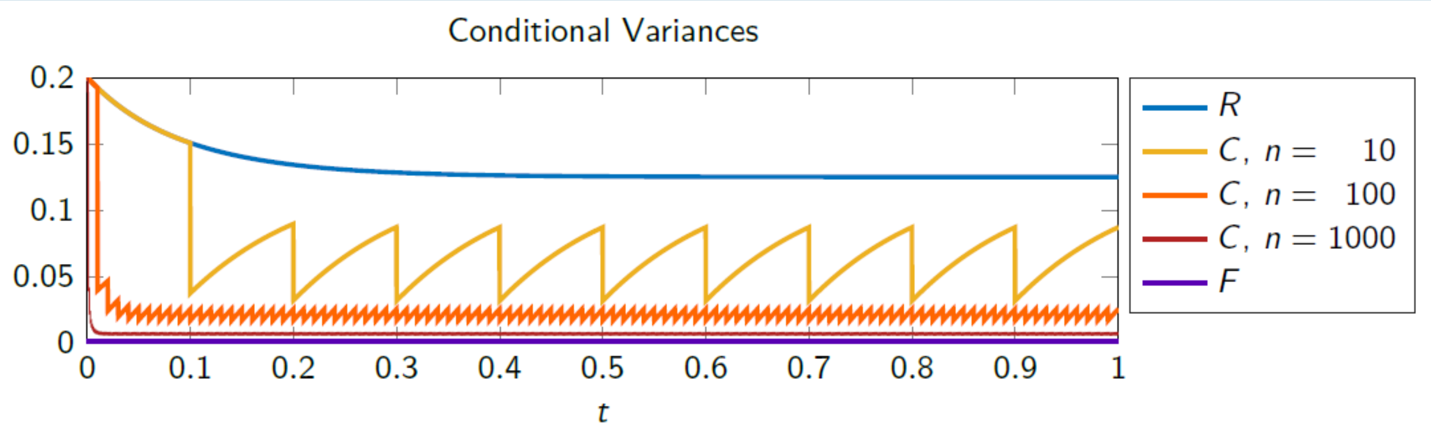
Experts with Bounded Variance

Convergence to full information

- Deterministic information dates, grid size Δ_n goes to zero
- Experts' covariances $\Gamma_k^{(n)}$ are bounded

➢ Convergence to full information



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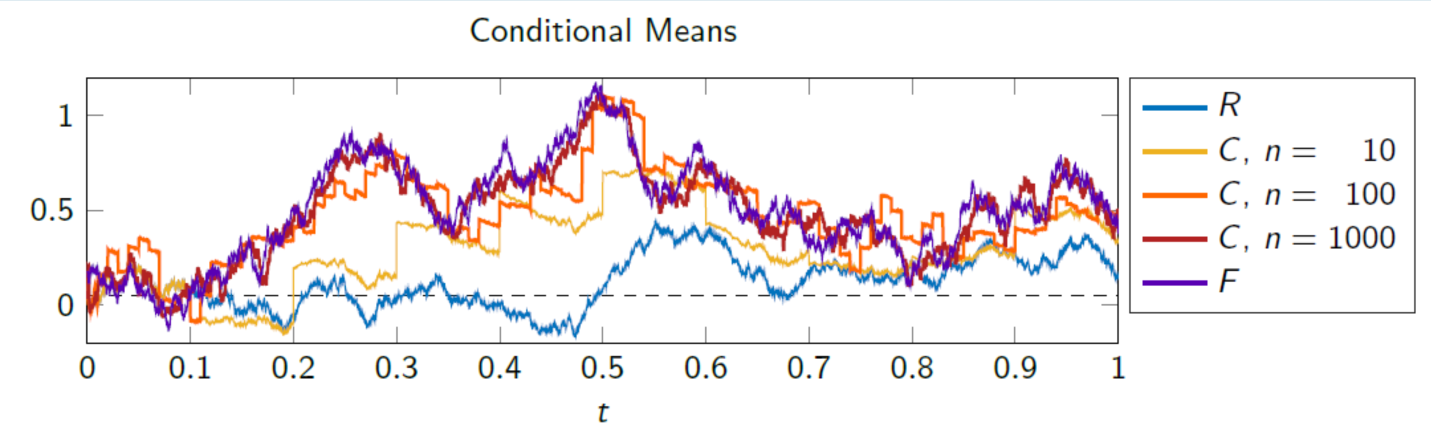

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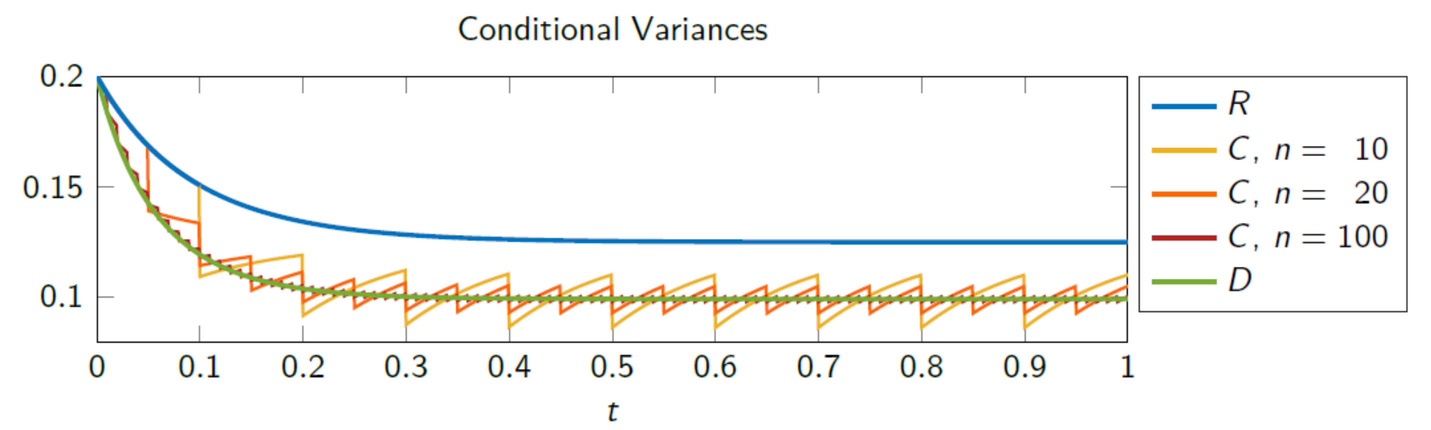
Diffusion Approximation

For deterministic information dates

- n deterministic equidistant information dates, grid size $\Delta_n = {}^{T}/_{n}$
- Experts' covariances grow linearly in n: $\Gamma_k^{(n)} = \frac{1}{\Delta_n}\sigma_J\sigma_J^\top$

➢ Convergence to "continuous-time expert"



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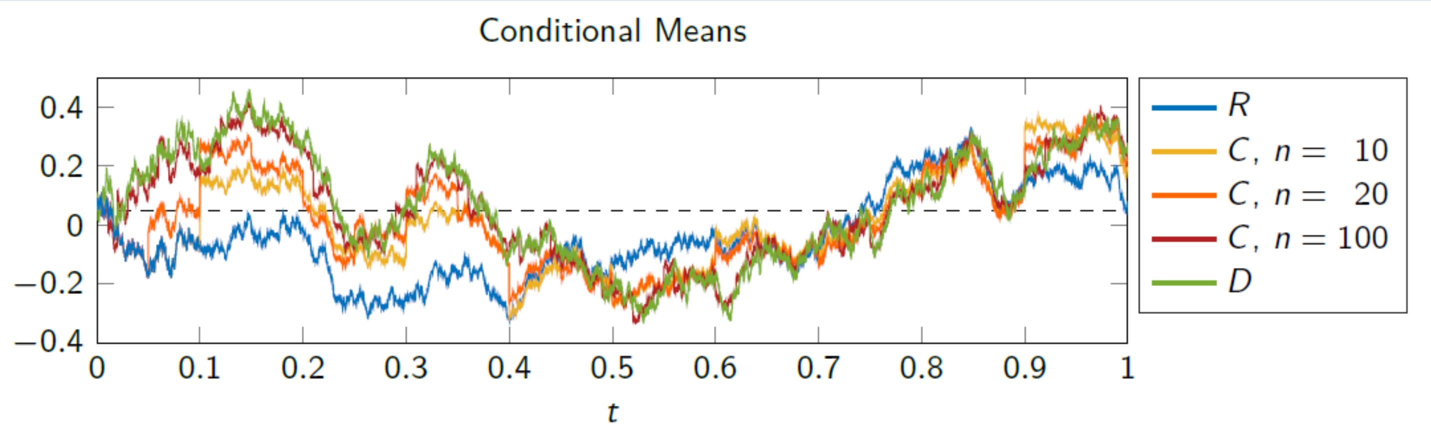

Diffusion Approximation

For random information dates

- T_k are jump times of a Poisson process with intensity λ
- Experts' covariances grow linearly in λ: $\Gamma_k^{(\lambda)} = \lambda \sigma_J \sigma_J^\top$

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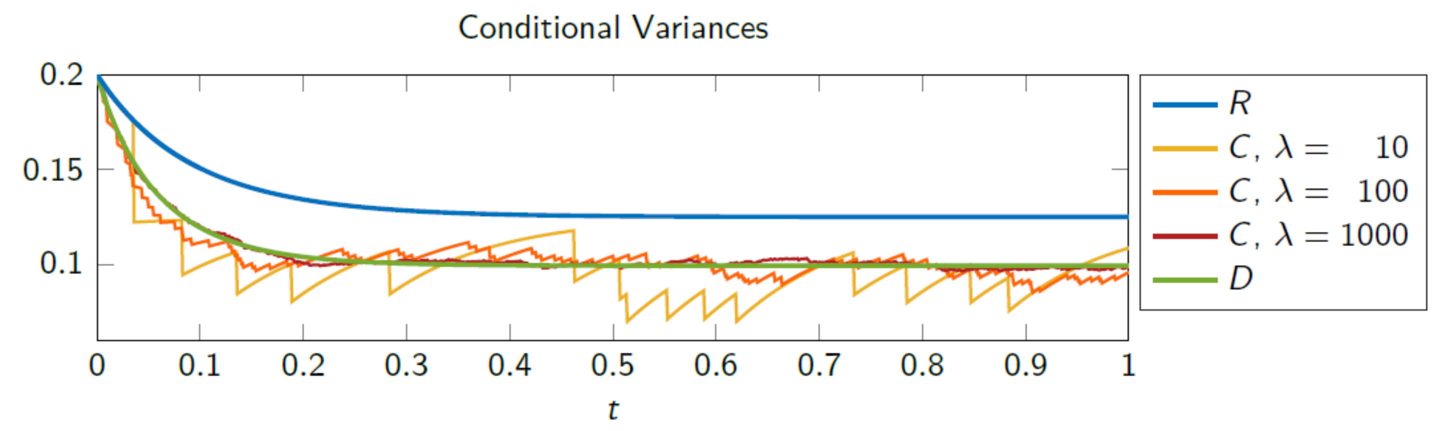

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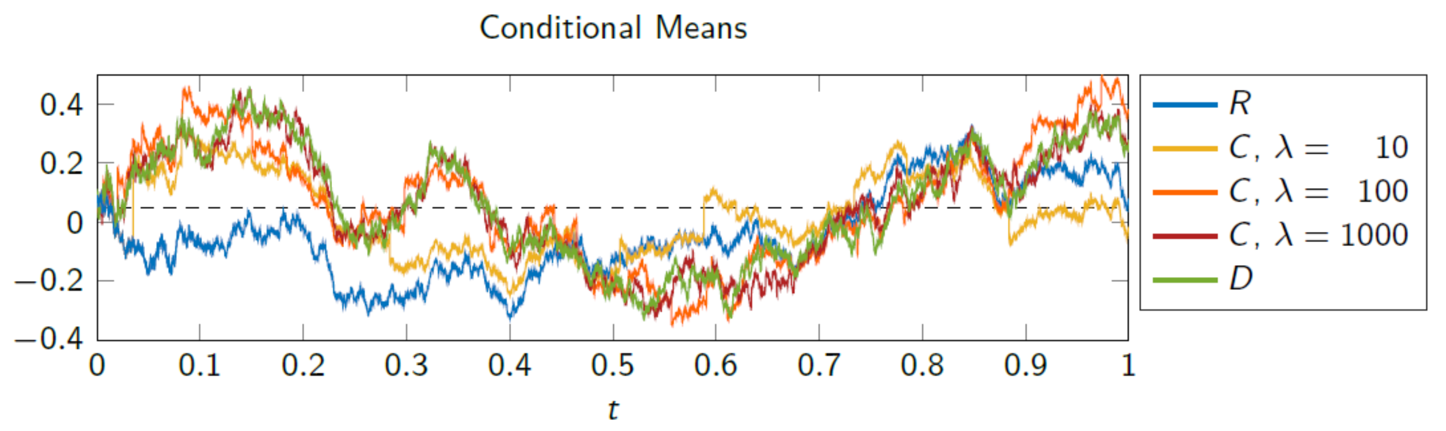


1. Model Uncertainty
2. Expert Opinions
3. **Robust Optimization with Expert Opinions**

Robust Optimization with Expert Opinions

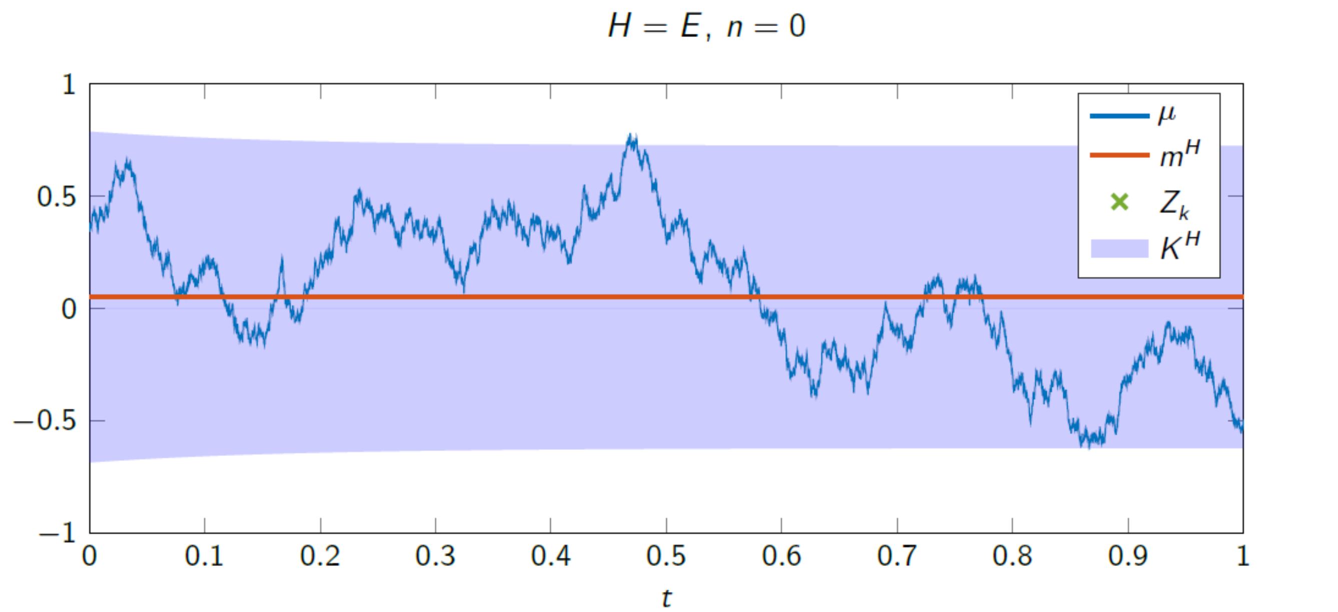
Combination of the two previous approaches

- Generalization of model uncertainty approach
- $(K_t)_{t\in[0,T]}$ motivated by filtering for drift estimation
- Leads to local optimization problems at each $t \in [0,T]$
- Decision is updated continuously in time

stochastic drift process → possible change in information → time-dependent uncertainty set

Examples of Time-Dependent Uncertainty Sets

Information from expert opinions



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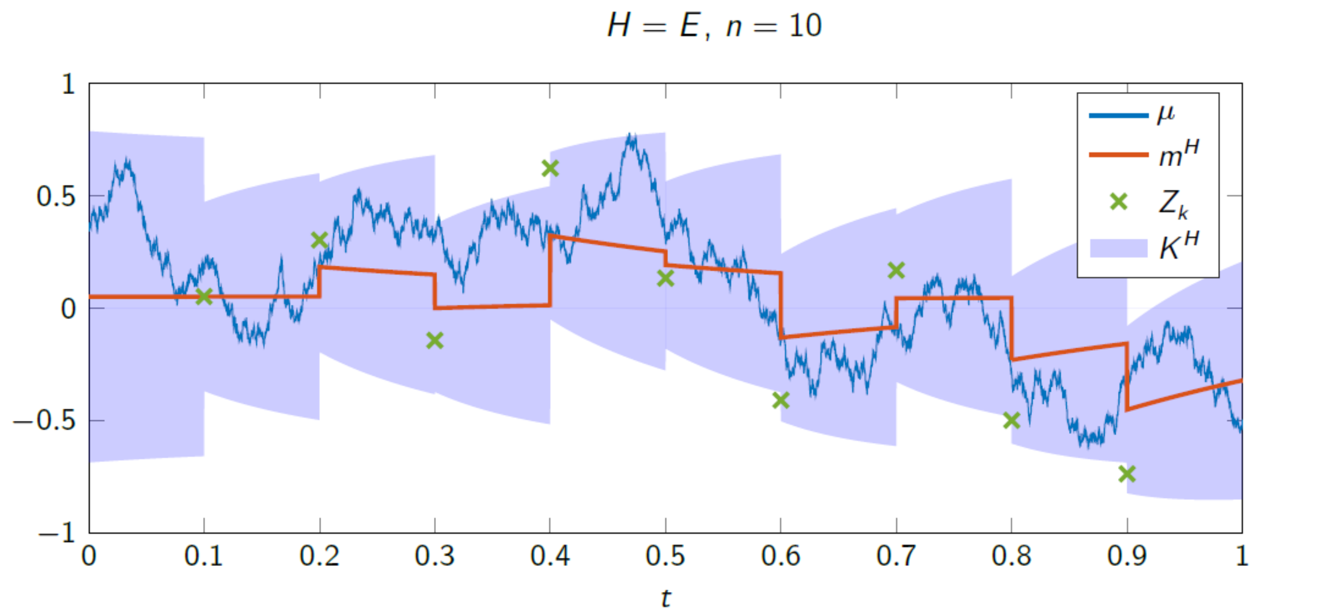

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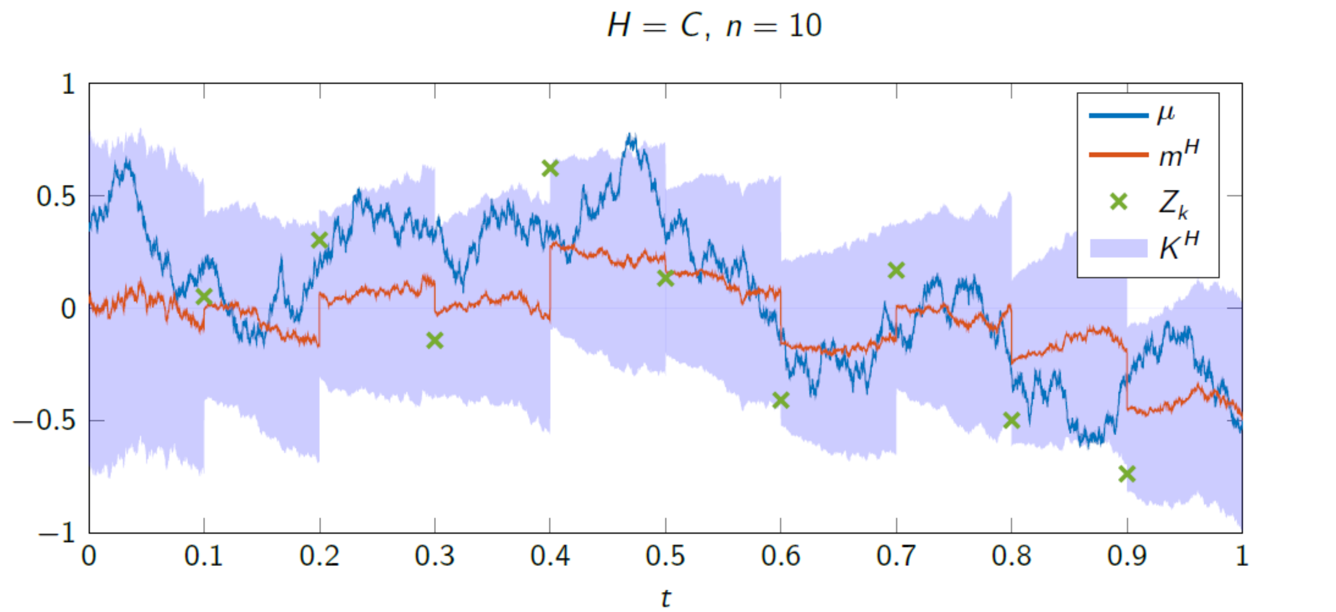

Summary

- Duality approach for solving a robust utility maximization problem
- Minimax theorem and convergence of the optimal strategy to a generalized uniform diversification strategy

- Expert opinions yield better estimates of the drift process
- Asymptotic results for large numbers of expert opinions

- Combination of two approaches for a financial market model with stochastic drift

Literature

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Questions or comments?
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