Utility Optimization and Indifference Pricing

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Utility maximization

General Problem Formulation

$$v(0,x,y) = \max_{\pi\in\Pi} \mathbb{E}\left[U\left(X_T^{(1)} + kg(y_T)\right)\right] \qquad \text{or} \qquad v(0,x,y) = \max_{\pi\in\Pi} \mathbb{E}\left[U\left(X_T^{(1/2)}\right)\right]$$

$$\begin{aligned} s.t. \quad dS_t &= \mu\, S_t\, dt + \sigma\, S_t\, dW_t^1 \\ dy_t &= adt + b(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2). \end{aligned}$$

$$dX_t^{(1)} = (rX_t^{(1)} + \pi_t X_t^{(1)}(\mu - r))dt + \theta_t X_t^{(1)}\sigma dW_t^1, \quad X_0 = x,$$

$$\text{or} \quad dX_t^{(2)} = (rX_t + \pi_t X_t^{(2)}(\mu - r))dt + \theta_t X_t^{(2)}\sigma dW_t^1 + y(t)dt, \quad X_0 = x$$

Utility Functions

- Power Utility:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

- Exponential Utility:

$$U(x) = -\frac{\exp(-\gamma x)}{\gamma}$$

- SAHARA Utility:

$$U(x) = \begin{cases} -\frac{1}{\alpha^2-1}\left(x + \sqrt{\beta^2+x^2}\right)^{-\alpha}\left(x + \alpha\sqrt{\beta^2+x^2}\right) & \alpha \neq 1 \\ \frac{1}{2}\ln(x + \sqrt{\beta^2+x^2}) + \frac{1}{2}\beta^{-2}x(\sqrt{\beta^2+x^2} - x) & \alpha = 1 \end{cases}$$

Applications

Indifference Pricing with SAHARA utility

- Power and exponential Utility have difficulties when pricing negative payouts

- Comparable results to power an exponential for extreme cases

- **Solve:**

$$\max_{\pi\in\Pi}\mathbb{E}\left[U(X_T^{(1)}+kg(y_T))\right]=\max_{\pi\in\Pi}\mathbb{E}\left[U\left(\widetilde{X}_T^{(1)}\right)\right]$$

- **Indifference Price:**

$$p=\widetilde{x}_0-x_0$$

Target Date Fund

- **Target Date Fund:**
  - Retirementproduct where you invest up to a certain target date
  - Investment strategy usually starts with 80-100% in Equity and ends with ca. 40% in equity (glidepaths)

- **Research Objective:**
  - Optimal glidepaths?
  - Effect of correlation between labour income and equity market?

- **Solve:**

$$\max_{\pi\in\Pi}\mathbb{E}\left[U\left(\widetilde{X}_T^{(2)}\right)\right]$$

HJB Formulation

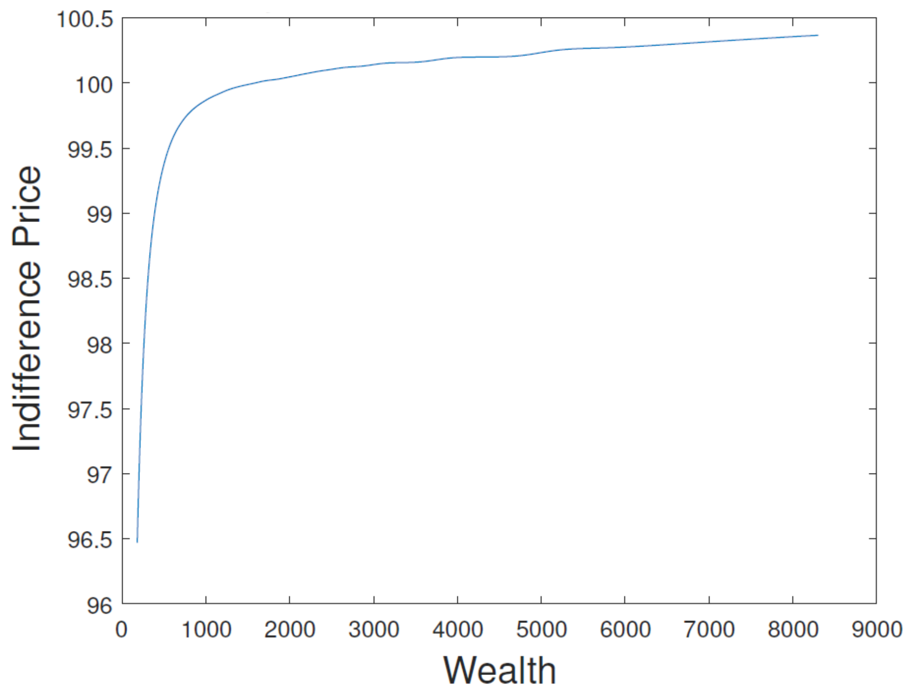
Two dimensional HJB equation

$$\max_{\pi}\left[u_t + (rx + (\mu - r)\pi x)u_x + a u_y + \frac{1}{2}(\sigma\pi x)^2 u_{xx} + \rho b\sigma\theta u_{xy} + \frac{1}{2}b^2 u_{yy}\right] = 0$$

- Optimal Strategy: $$\pi^*(t,x,y) = \frac{(\mu - r)/(x\sigma^2)}{-u_{xx}/u_x} + \frac{\rho b/(x\sigma)}{-u_{xx}/u_{xy}}.$$

Note: The HJB Equation may be reduced by one dimension

Indifference Pricing with SAHARA utility



Target Date Funds