# Utility Indifference Pricing Methods for Incomplete Markets

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# Motivation

- Pricing in incomplete market of new asset: utility indifference pricing principle
- Requires that we solve two (nonlinear) optimization problems:

Optimized expected utility of terminal wealth including asset

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Optimized expected utility of terminal wealth without asset

- Hardly any closed-form solutions. See for example (Duffie et al. 1997, Cvitanic et al. 2001, Musiela & Zariphopoulou, 2004, Carmona, 2009).

# Preferences

Closed-form solutions that exist in continuous time for optimal portfolio problem are often based on

- invariance in optimized preference structure and
- suitable stochastic dynamics.

Example:

CRRA utility on $\mathbb{R}^+$ (Merton, 1969): for $0 < \gamma \neq 1$

$$
\begin{aligned}
V_T(w) &= U_\gamma(w) := \frac{w^{1-\gamma}}{1-\gamma} \\
\Rightarrow V_t(w) &= a(t) U_\gamma(w).
\end{aligned}
$$

# Motivation

- Goal: to calculate exact optimal investment strategies for
  - discrete time Markov process and
  - simple preference structures

  that are 'close to' a given specification.
- Risk aversion parameter and parameters for dynamics of risk factors are difficult to measure exactly anyway.

# Asset Dynamics

We use the following model:

- Discrete Markov dynamics $(t, S_t, Y_t)$ for risky asset $S$ and other (untradeable) risk factors $Y$ on finite state space

$$\mathcal{T} = \bigcup_{m=0}^{n} \mathcal{T}_m, \qquad \mathcal{T}_m = \bigcup_{k=0}^{m} \bigcup_{j=0}^{m} \{(m\Delta t, S_k^m, Y_j^m)\},$$

  with transition probabilities which ensure that $(t, S_t, Y_t) \in \mathcal{T}$.
- In our examples we take $\dim(Y) = 1$ and two possible successor values for each $S_k^m$ and $Y_l^m$.

# Preferences

Utility for terminal wealth (with and without contingent claim) restricted to class $\mathcal{H}$ of functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfying

- $f$ is closed proper concave, so epigraph

$$\mathrm{Epi}_f = \{(x, y) \mid y \leq f(x)\}$$

  is a closed convex subset of $\mathbb{R}^2$, and
- on its effective domain $E_f = \{x \in \mathbb{R} : f(x) > -\infty\}$, $f$ is piecewise linear and the number of points where $f$ is not differentiable is finite on $E_f$, and
- there exists an $\bar{x} \in \mathbb{R}$ such that $f(x)$ is a real constant for all $x \geq \bar{x}$.

Interior left- and righthand side derivatives exist and are monotone.

# Preferences

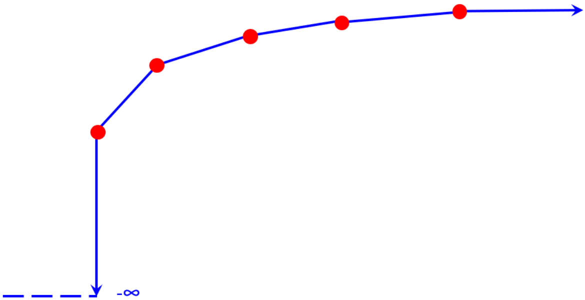



Figure: Example of a function in class class $\mathcal{H}$.

# Optimization Problem

For a given utility function $U \in \mathcal{H}$, we look for strategy $\phi$ maximizing expected utility of terminal wealth at $T = n\Delta t$:

$$\max_{\phi \in \Phi} \mathbb{E}[U(W_T^{\phi})] \tag{1}$$

subject to

$$W_{t+\Delta t}^{\phi} \;=\; \phi_t(X_t)\frac{S_{t+\Delta t}}{S_t} + (W_t^{\phi} - \phi_t(X_t))\, R(t) \tag{2}$$

where strategy $\phi$ for investment in risky asset can depend on state $X_t = (S_t, Y_t, W_t)$, $R$ is deterministic risk-free rate, and $W_0^{\phi} = w_0$ the initial wealth.

# Indifference pricing

Value function for problem without and with contingent claim paying $\Psi(X_T)$:

$$\begin{aligned} V_{t,S_t,Y_t}(W_t) &= \max_{\phi_t,\phi_{t+\Delta t},\dots,\phi_T} \mathbb{E}[\, U(W_T^{\phi}) \mid X_t \,] \\ \tilde{V}_{t,S_t,Y_t}(W_t) &= \max_{\tilde{\phi}_t,\tilde{\phi}_{t+\Delta t},\dots,\tilde{\phi}_T} \mathbb{E}[\, U(W_T^{\tilde{\phi}} - \Psi(X_T)) \mid X_t \,] \end{aligned}$$

Selling price $\pi_\Psi(w_0)$ of claim $\Psi$ must satisfy

$$V_{0,S_0,Y_0}(w_0) = \tilde{V}_{0,S_0,Y_0}(w_0 + \pi_\Psi(w_0))$$

Structure we exploit: for every possible value of $(t, S_t, Y_t)$, optimal strategy only depends on wealth, so we write

$$\phi_t(X_t) = \beta_{t,S_t,Y_t}(W_t).$$

# Main Result

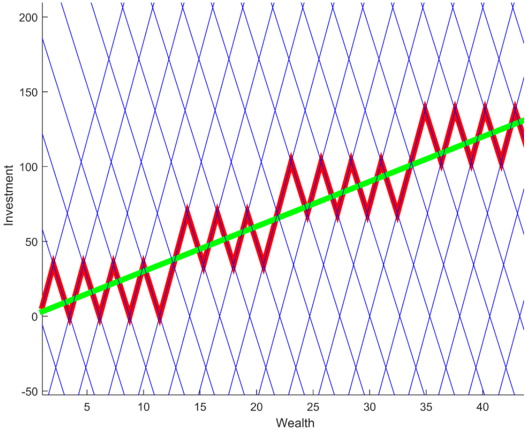
Invariance: dynamic programming principle maps $\mathcal{H}$ on itself!

## Theorem

*Assume* $V_{t,S,Y} \in \mathcal{H}$ *for all* $(t, S, Y) \in \mathcal{T}_{m+1}$. *Then*

- $V_{t,S,Y} \in \mathcal{H}$ *for all* $(t, S, Y) \in \mathcal{T}_m$.
- *Functions* $w \to \beta_{t,S,Y}(w)$ *and* $w \to V_{t,S,Y}(w)$ *are continuous on their domain for all* $(t, S, Y) \in \mathcal{T}_m$.
- *Graph of optimal strategy* $\{(w, \beta_{t,S,Y}(w)) : w \in \mathbb{R}\} \subset \mathbb{R}^2$ *must lie on grid of 2 sets of parallel lines that can be pre-computed.*

# Optimal strategy is subset of known grid



# Main Result

### Theorem

*Assume that $V_{t+\Delta t,uS,\star}$ and $V_{t+\Delta t,dS,\star}$ have singular values $x_1^u < x_2^u < \ldots < x_{N_u}^u$ and $x_1^d, x_2^d < \ldots < x_{N_d}^d$ respectively, and let*

$$z_i^u = V'^{-}_{t+\Delta t,uS,\star}(x_i^u), \quad z_i^d = V'^{-}_{t+\Delta t,dS,\star}(x_i^d).$$

*Then $x_k$ (the singular values of $V_{t,S,Y}$ in reverse order), the optimal strategies $\beta_k = \beta_{t,S,Y}(x_k)$ and the left-hand side derivatives $z_k = V'^{-}_{t,S,Y}(x_k)$ satisfy, for $k \geq 1$,*

$$\begin{aligned}
x_k &= R^{-1}(qx^u_{i_k} + (1-q)x^d_{j_k}), & \beta_k &= (x^u_{i_k} - x^d_{j_k})/(u-d), \\
z_u &= \tfrac{p}{q}\, z^u_{i_k}, & z_d &= \tfrac{1-p}{1-q}\, z^d_{j_k}, \\
z_k &= z_u \wedge z_d, & v_k &= v_{k-1} - z_{k-1}(x_k - x_{k-1}), \\
i^u_{k+1} &= i^u_k - \mathbf{1}_{\{z_k = z_u\}}, & i^d_{k+1} &= i^d_k - \mathbf{1}_{\{z_k = z_d\}}.
\end{aligned}$$

# Efficient Indifference Pricing

Consequence:

- Fast numerical algorithms: sequence of binary choices
- On finite state space in finite time we generate exact solutions for value functions, which must be in $\mathcal{H}$
- But increasing concave functions in $\mathcal{H}$ also allow "deleting" of singular points which reduces computational cost for a priori given error bound: gives $\epsilon$-close strategies.

This makes calculation of utility indifference prices relatively easy.

# Approximation in $\mathcal{H}$

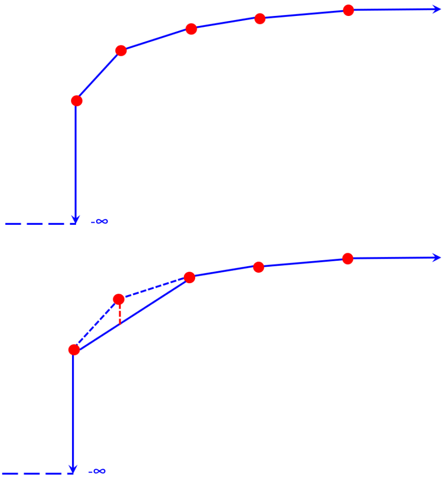



Figure: Example of a function in class $\mathcal{H}$ and its approximation.

# Convergence to viscosity solution

- To find approximations for investment strategies in continuous time problems we consider the HJB equation with (general) terminal utility $\bar{U}$ and value function $\bar{V}_{t,S}(w)$:

$$\begin{aligned} \bar{V}_{T,S}(w) &= \bar{U}(w - \Phi(S)) \\ 0 &= \partial_t \bar{V} + \mu S \, \partial_S \bar{V} + \tfrac{1}{2}\sigma^2 S^2 \, \partial_{SS} \bar{V} \\ &\quad + \max_{\pi \in \mathbb{R}} \left[ (r + \pi(\mu - r)) w \, \partial_w \bar{V} + \tfrac{1}{2}\sigma^2 w^2 \pi^2 \right] \end{aligned}$$

  for a positive, bounded and globally Lipschitz continuous function $\Phi$ ($Y$ is taken constant here).
- We assume that

$$(\forall w \geq \bar{w}) \qquad 0 \leq \bar{U}(w) - \bar{u} \leq \bar{c}(w - \bar{w})^{1-\bar{\gamma}}$$

  for appropriate constants $\bar{u}$ $\bar{c}$, $\bar{w}$, $\bar{\gamma}$, which ensures solutions exist.

# Convergence to viscosity solution

We create sequence of approximations with $n \in \mathbb{N}^*$ equidistant timesteps, $n^\nu$ terminal singular points at distance $n^{-\zeta}$ with $\nu > \zeta > 0$, and denote solutions by $\bar{V}^n_{t,S}(w)$ for all $(t,S,w) \in \mathcal{T}^n \times \mathbb{R}$.

## Theorem

*If the HJB equation has a unique viscosity solution $\bar{V}$ that is increasing and concave for $(t,S,w) \in \mathcal{Z} = [0,T] \times [0,\infty) \times [\bar{w},\infty)$, then we have for all $(t,S,w) \in \mathcal{Z}$:*

$$\lim_{\substack{n\to\infty \\ \mathcal{T}^n\times\mathbb{R}\ni(t_n,s_n,w_n)\to(t,S,w)}} \bar{V}^n_{t_n,s_n}(w_n) = \bar{V}_{t,S}(w).$$

# Exponential Indifference pricing

Model of (Musiela & Zariphopoulou, 2004) in continuous time

$$\begin{aligned} dS_t &= \mu_S S_t dt + \sigma_S S_t dW_t^S, \\ dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, \qquad d\langle W^S, W^Y \rangle_t = \rho dt, \end{aligned}$$

with only $S$ tradeable.

Indifference price for contingent claim $\Psi(Y_T)$ under preferences $U(x) = -e^{-\gamma x}$ can be shown to be

$$\pi^{\text{cont}} = \frac{\ln \mathbb{E}[\, e^{\gamma(1-\rho^2)\Psi(Y_T)} \frac{d\mathbb{Q}}{d\mathbb{P}} \,]}{\gamma(1-\rho^2)}, \qquad \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma} W_T^S - \frac{\mu^2}{2\sigma^2} T}.$$

# Exponential Indifference pricing

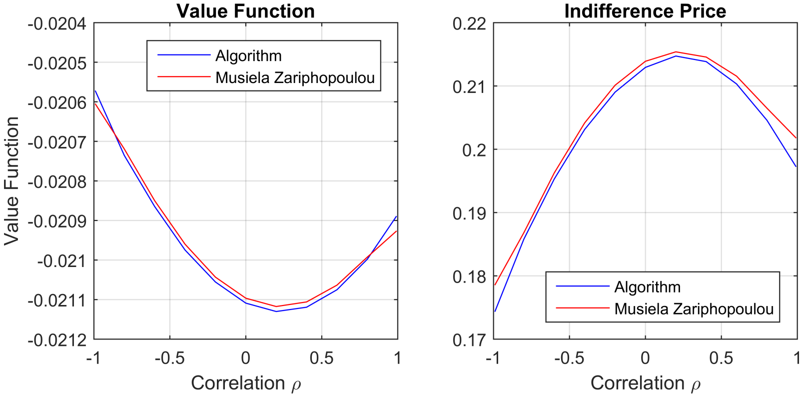



Figure: Parameters $S_0 = K = 5$, $\sigma = 10\%$, rest as before.

# Optimal Strategies under Stochastic Volatility

Model (Heston, 1993) in continuous time:

$$
\begin{aligned}
dY_t &= \kappa(\theta - Y_t)dt \; + \; \omega\sqrt{Y_t}dW_t^S, \\
dS_t &= (r + \lambda Y_t)dt \; + \; \sqrt{Y_t}dW_t^Y, \qquad d\langle W^S, W^Y\rangle_t = \rho dt,
\end{aligned}
$$

with $\lambda$ the market price of volatility risk parameter.
Optimal strategy under CRRA with parameter $\gamma$ is (Kraft, 2005)

$$
\beta^{\text{cont}}_{t,S_t,Y_t}(w) \; = \; w\,\frac{(e^{a(T-t)} - 1)(\lambda + \gamma^{-1}(1-\gamma)\rho\sigma\lambda^2)}{\gamma(e^{a(T-t)}(k+a) + a - k)}
$$

$$
k = \kappa - \tfrac{\rho\lambda\sigma}{1-\gamma^{-1}}, \; c = \tfrac{\gamma}{\gamma+\rho^2(1-\gamma)}, \; B = -\tfrac{\lambda^2(1-\gamma)}{2c\gamma}, \; a = \sqrt{k^2 + 2B\sigma^2}.
$$

# Approximation in state space

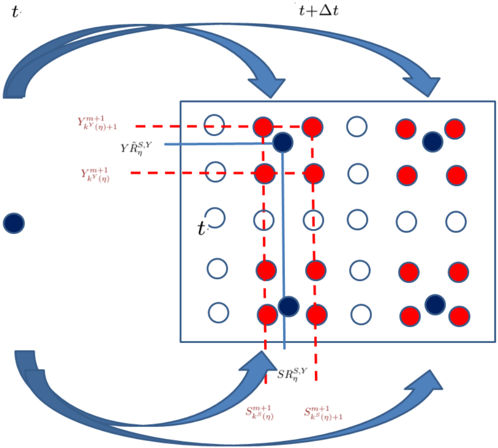



Figure: Approximation on grid implemented by "splitting" probabilities.

# Nonlinear Strategies

SAHARA preferences (Chen, Pelsser & Vellekoop, 2011) :

$$U_{\gamma,\beta}(w) \;=\; \frac{1}{1-\gamma^2}\left(w + \sqrt{\beta^2 + w^2}\right)^{-\gamma}\left(w + \gamma\sqrt{\beta^2 + w^2}\right)$$

in standard Black-Scholes economy.
Optimal strategy:

$$\beta^{\mathrm{cont}}_{t,S_t}(w) \;=\; \frac{\mu_S - r}{\gamma\sigma_S^2}\sqrt{w^2 + \beta^2 e^{-2(r-\frac{1}{2}((\mu_S-r)/(\gamma\sigma_S))^2)(T-t)}},$$

shows 'gambling for resurrection'.
Risk aversion is always positive but not monotone.

# Approximations for continuous time problems

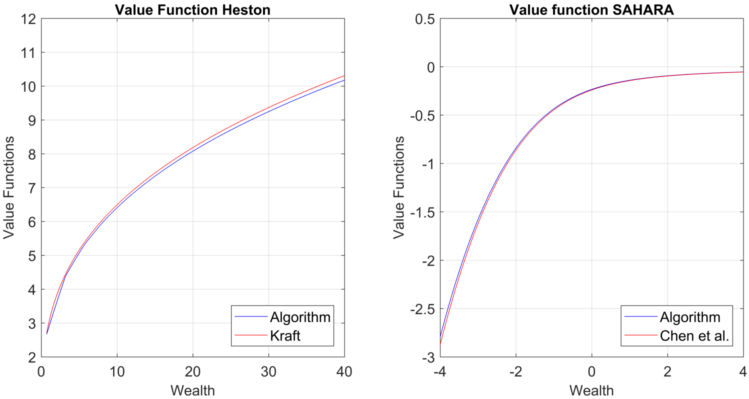



Figure: Parameters $r = 10\%$, $\sqrt{Y_0} = 25\%$, $\rho = 10\%$, $\omega = 39\%$, $\kappa = 1.15$, $\theta = 16\%$, $\lambda = \frac{1}{3}$, $\gamma = \frac{2}{3}$, $\alpha = 2$, $\beta = 2.66$.

# Conclusions and Further Research

- Calculation of exact optimal investment strategies in discretized models is possible for $\epsilon$-close preferences
- Good approximation of optimal value function does not necessitate smooth approximation of optimal investment strategy
- Algorithms are simple; mathematical work is in proving continuity of optimal strategy on the grid and convergence to viscosity solutions.

Carmona, R., editor (2009).
*Indifference Pricing: Theory and Applications*.
Princeton University Press.

Chen, A., Pelsser, A., and Vellekoop, M. (2011).
Modeling non-monotone risk aversion using SAHARA utility functions.
*Journal of Economic Theory*, 146(5):2075 – 2092.

Cvitanić, J., Schachermayer, W., and Wang, H. (2001).
Utility maximization in incomplete markets with random endowment.
*Finance Stoch.*, 5(2):259–272.

Duffie, D., Fleming, W., Soner, H. M., and Zariphopoulou, T. (1997).
Hedging in incomplete markets with HARA utility.
*Journal of Economic Dynamics and Control*, 21(4-5):753–782.

Gaudenzi, M., Lepellere, M., and Zanette, A. (2010).
The singular points method for pricing American path-dependent options.
*Journal of Computational Finance*, 14:1460–1559.

Kraft, H. (2005).
Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility.
*Quantitative Finance*, 5:303–313.

Merton, R. (1969).
Lifetime portfolio selection under uncertainty: the continuous-time case.
*Review of Economics and Statistics*, 51(3):247–257.

Musiela, M. and Zariphopoulou, T. (2004).
An example of indifference prices under exponential preferences.
*Finance and Stochastics*, 8(2):229–239.