Stochastic Interest Rate Modeling in a Low Interest Rate Environment

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DGVFM
DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.

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Agenda

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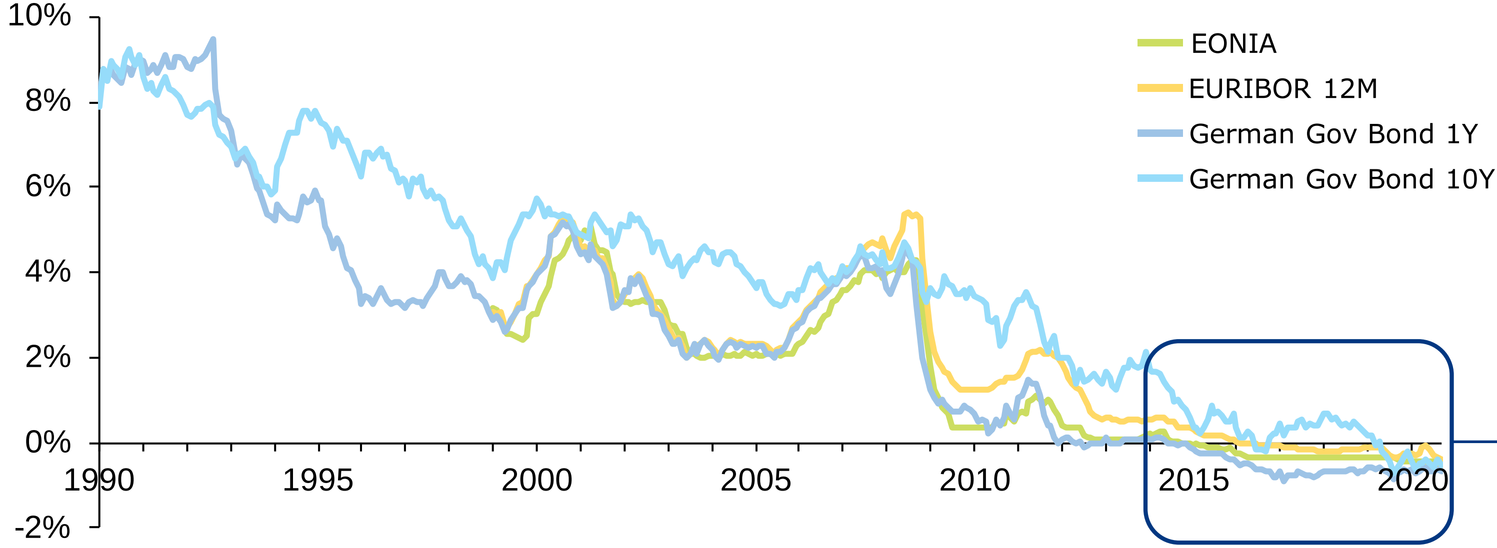
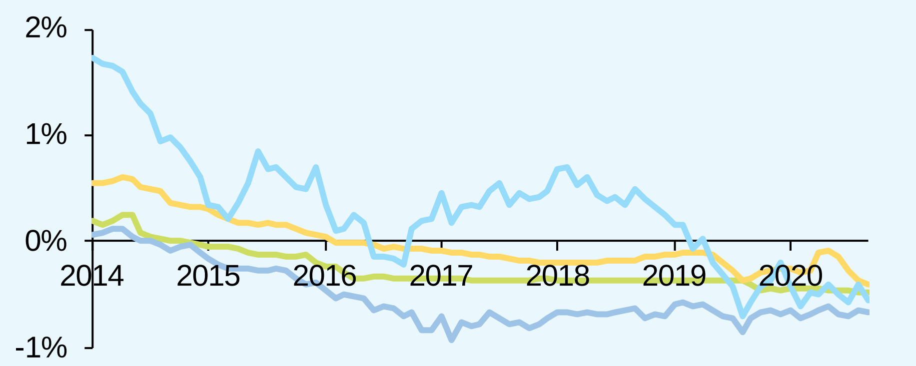
Historical interest rates



Interest rates on a low level in the last years



Source: German Federal Bank

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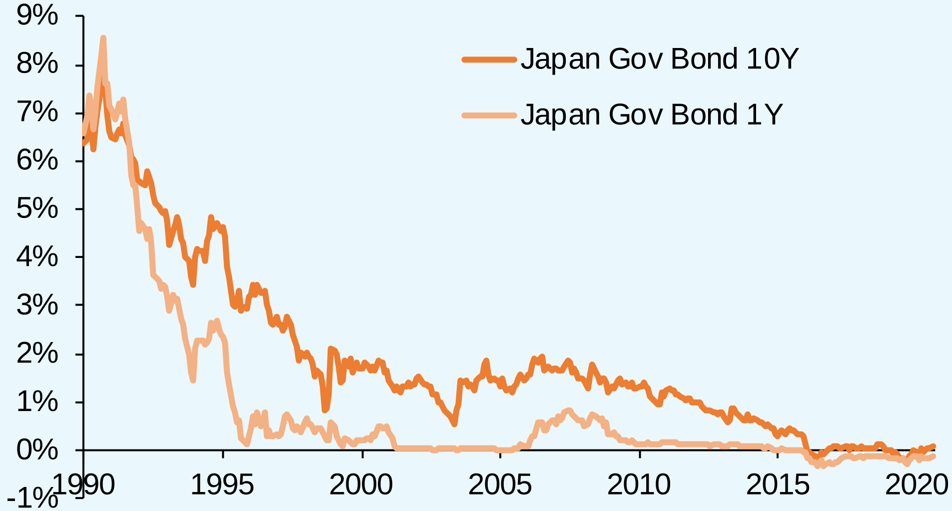
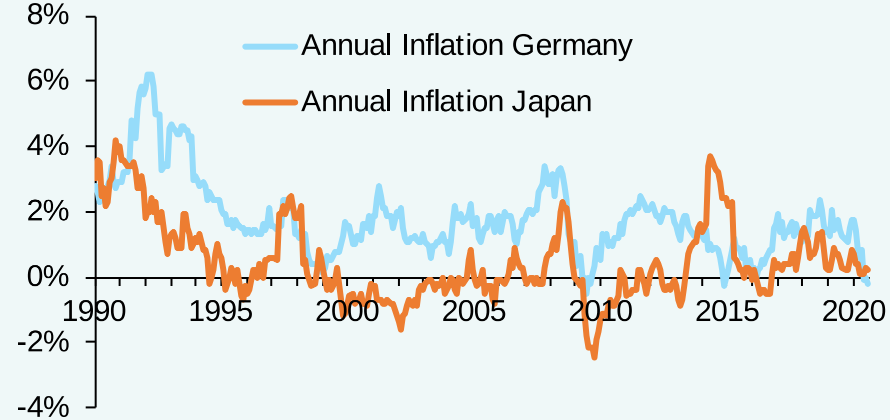
Digression: Japan - economy with long period of low interest rates

Low interest rates for long period…



… but differences in other economic indicators



Source: Ministry of Finance Japan, IMF, eurostat

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Spot rate, short rate and yield curve

Spot rate $R(t,T)$

Spot interest rate $R(t,T)$

at time t for maturity T

Zero-coupon-bond price $P(t,T)$

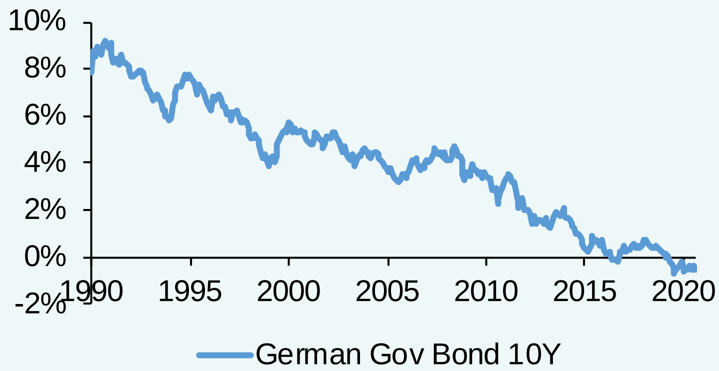
$$P(t,T) = e^{-R(t,T)\cdot(T-t)}$$

Short rate $r(t)$

$$r(t) = \lim_{T\to t^+} R(t,T)$$

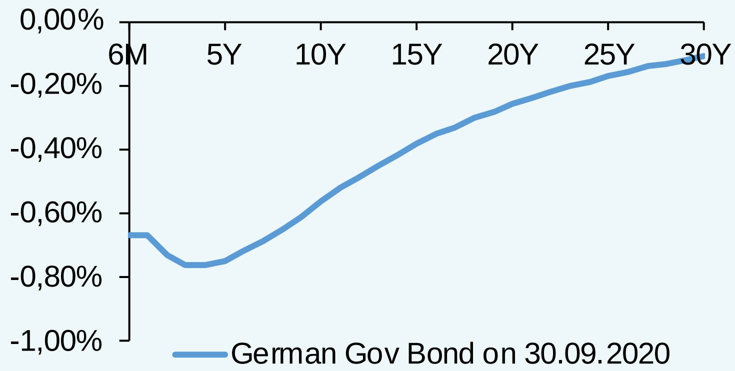
Spot rates for fixed maturity T

Fixed maturity $T = 10$ years, different times t



Yield curve: spot rates for fixed time t

Fixed time t = 30.09.2020, different maturities T



Vasicek model

Short rate dynamics

$$dr(t) = k\big(\theta - r(t)\big)dt + \sigma dW(t)$$

with

- k - speed of mean reversion
- θ - long term mean level
- σ - instantaneous volatility
- W - Brownian motion

Properties

Mean reverting

if $r(t) > \theta \Rightarrow \big(\theta - r(t)\big) < 0$ (the interest rate decreases over time)

if $r(t) < \theta \Rightarrow \big(\theta - r(t)\big) > 0$ (the interest rate increases over time)

Affine term-structure model

$P(t,T) = A(t,T) \cdot e^{-B(t,T) \cdot r(t)}$

with A, B deterministic functions of time, depending on k, θ, σ

Short rate models*

1973 1977 1978 1985 1986 1990 1991

Merton (1973)

$$d\,r(t) = \boldsymbol{\alpha dt} + \sigma dW(t)$$

Vasicek (1977)

$$d\,r(t) = \boldsymbol{k}\big(\boldsymbol{\theta} - \boldsymbol{r(t)}\big)dt + \sigma dW(t)$$

Dothan (1978)

$$d\,r(t) = \boldsymbol{a} \cdot \boldsymbol{r(t)dt} + \sigma dW(t)$$

Exponential Vasicek

$$dy(t) = \big(\theta - a \cdot y(t)\big)dt + \sigma dW(t)$$
$$\boldsymbol{r(t)} = \mathbf{exp}\big(\boldsymbol{y(t)}\big)$$

Cox-Ingersoll-Ross (1985)

$$d\,r(t) = k\big(\theta - r(t)\big)dt + \sigma\sqrt{\boldsymbol{r(t)}}dW(t)$$

Ho-Lee (1986)

$$d\,r(t) = \boldsymbol{\theta(t)}dt + \sigma dW(t)$$

Hull-White (1990)

$$d\,r(t) = \big(\boldsymbol{\theta(t)} - \boldsymbol{a(t)} \cdot r(t)\big)dt + \boldsymbol{\sigma(t)}dW(t)$$

Black-Karasinski (1991)

$$d\,\mathbf{log}\big(\boldsymbol{r(t)}\big) = \big(\theta(t) - a(t) \cdot \mathbf{log}\big(\boldsymbol{r(t)}\big)\big)dt + \sigma(t)dW(t)$$

* Selection of one-factor short rate models based on Brigo and Mercurio (2006)

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Vasicek: calibration to historical data

Parameters

Short rate dynamics $dr(t) = k\big(\theta - r(t)\big)dt + \sigma dW(t)$

Aim: calibrate parameters k, θ, σ to fit observed historical data.

Calibration: Maximum Likelihood Estimation

„choose parameter k, θ, σ, such that the observed data is most probable"

Given $n+1$ short rates $r(t_0), \dots, r(t_n)$, the joint density function is

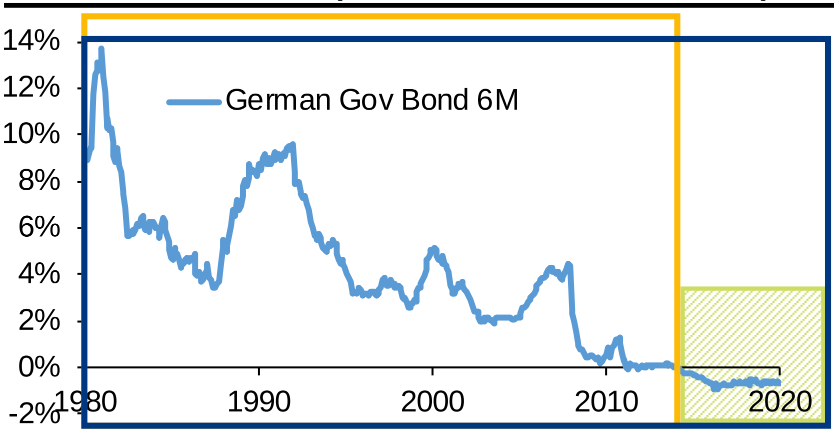
$$L(k,\theta,\sigma) = p\big(r(t_0)\big) \prod_{i=1}^{n} p(r(t_i)\,|r(t_{i-1}); k,\theta,\sigma)\,.$$

Chose parameters to maximize L: $(k,\theta,\sigma) = \arg\max_{k,\theta,\sigma} L(k,\theta,\sigma)$

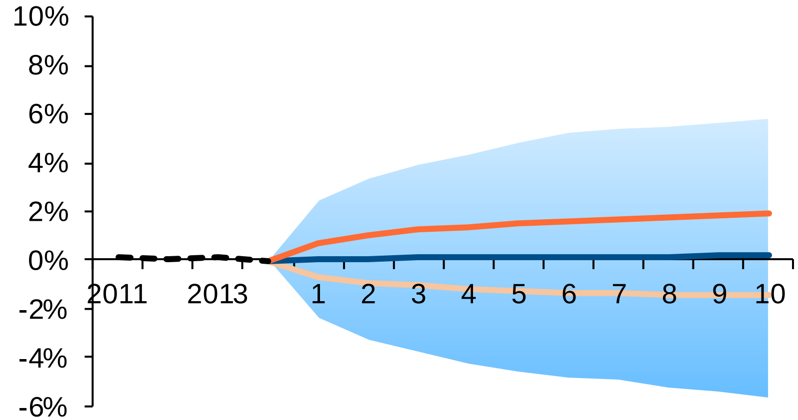
used data: 6-month German Government Bond Yield
(alternatives: EONIA, 3-month EURIBOR)

Vasicek results (German Gov Bond 6M)

Historical data (German Gov Bond 6M)



Calibration period 09/1980 – 09/2014



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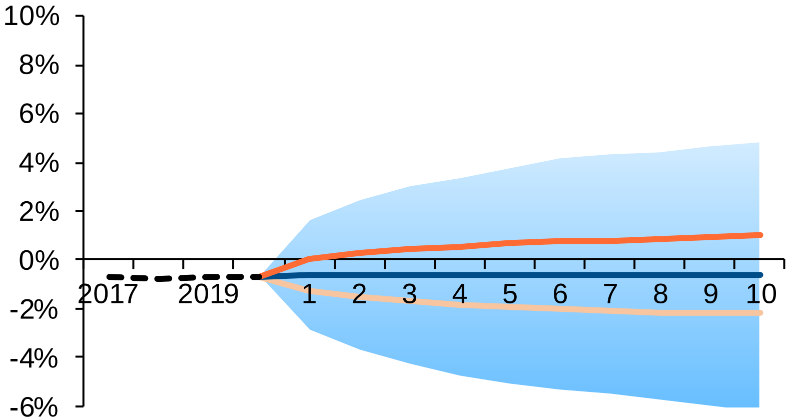

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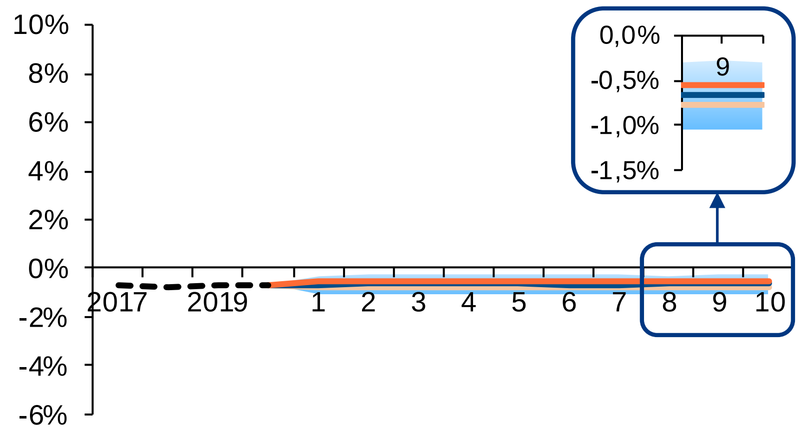


History 1%-99% Quantile 25% Quantile 50% Quantile 75% Quantile

Cox-Ingersoll-Ross

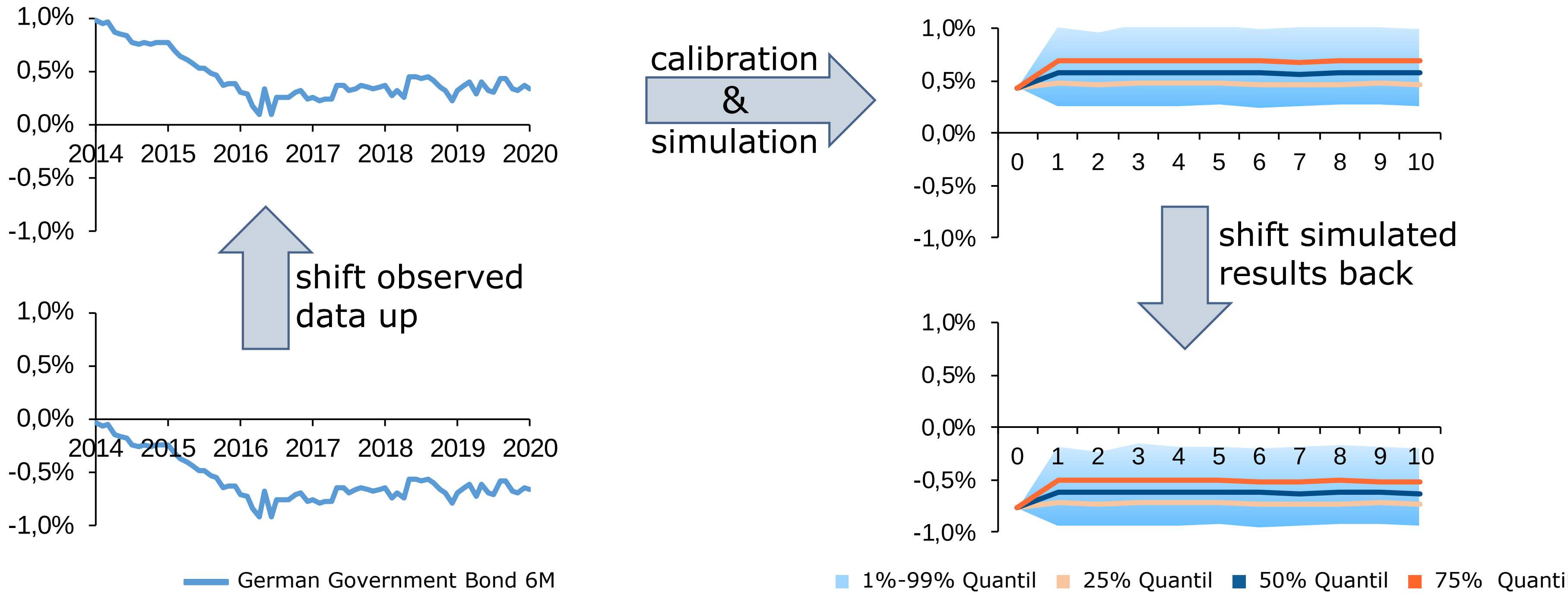
Short rate dynamics

$$dr(t) = k\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}dW(t)$$

➡ Gives positive interest rates

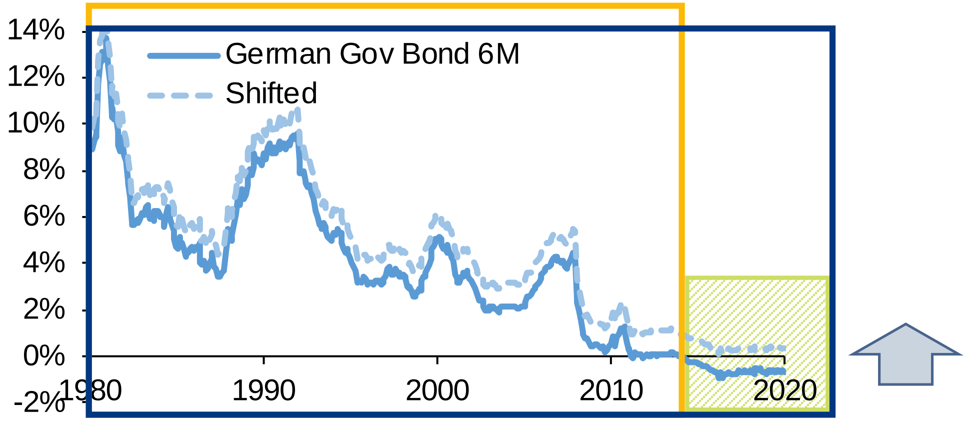
Calibration to negative interest rates: shift observed data

(cf. Orlando et. al., 2019)



Cox-Ingersoll-Ross results (German Gov Bond 6M)

Historical data (German Gov Bond 6M)



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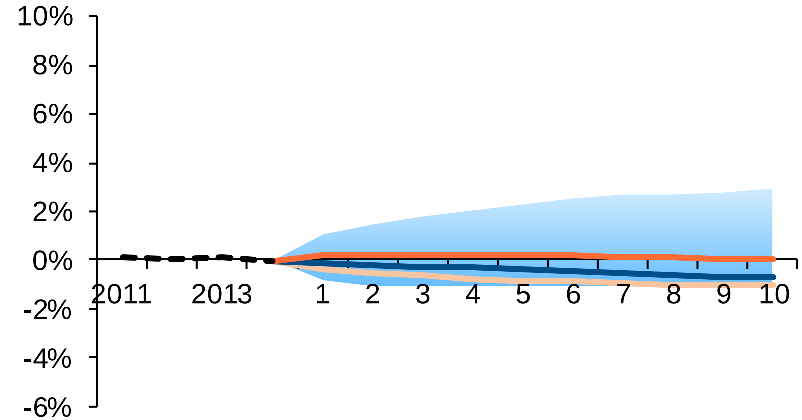

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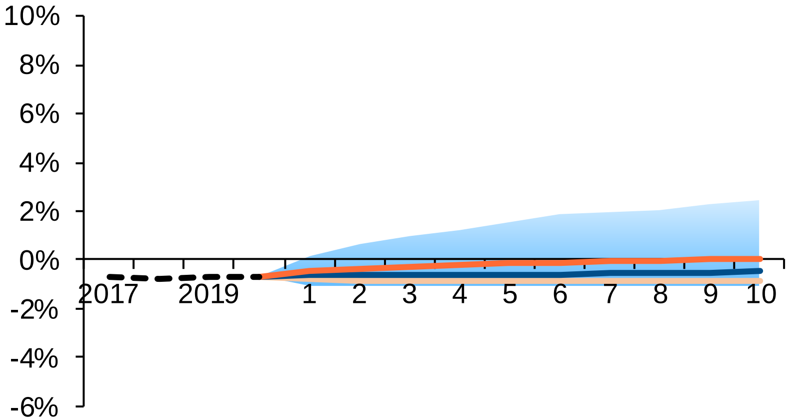
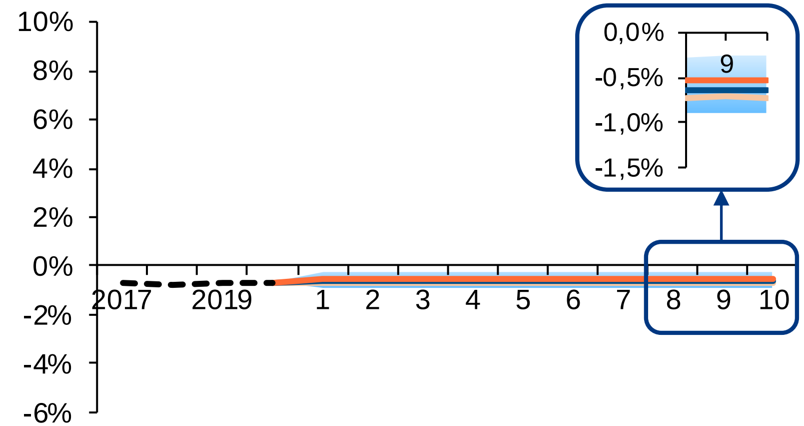


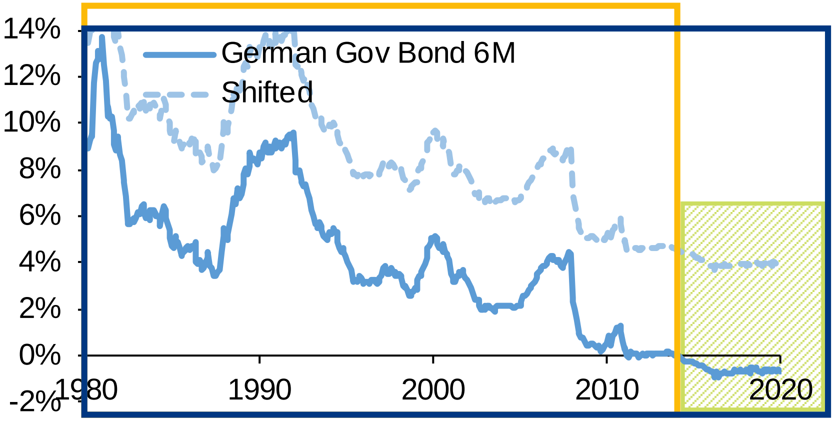
Calibration period 09/2014 – 09/2020



History 1%-99% Quantile 25% Quantile 50% Quantile 75% Quantile

Cox-Ingersoll-Ross results (German Gov Bond 6M)

Historical data (German Gov Bond 6M)



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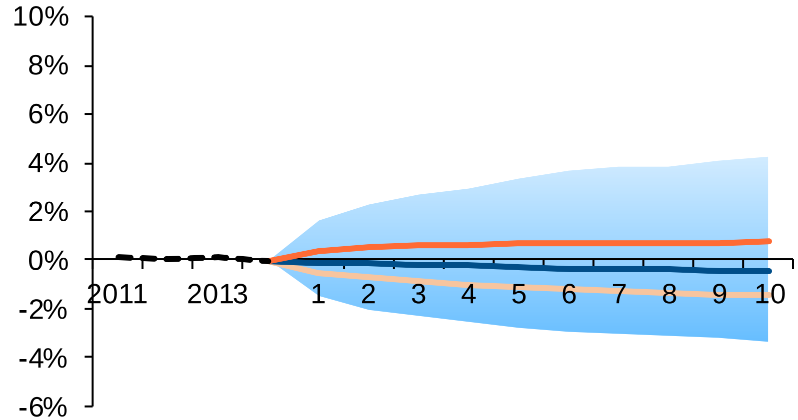

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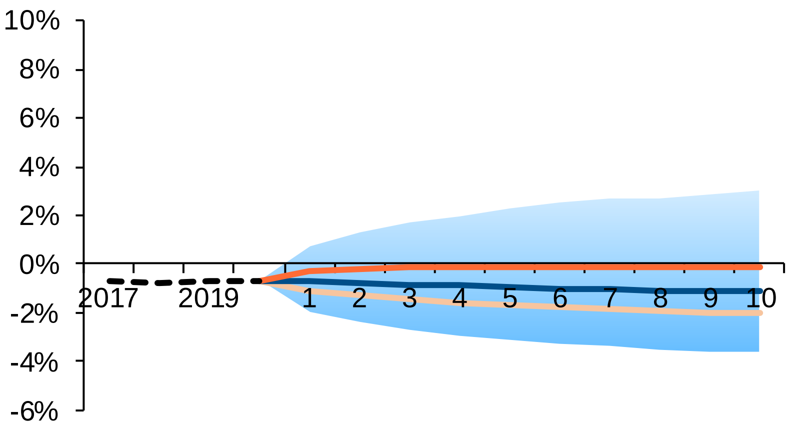
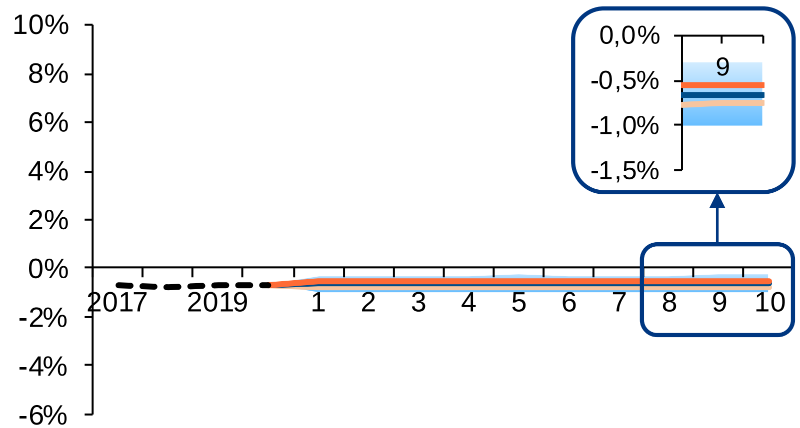


Calibration period 09/2014 – 09/2020



History   1%-99% Quantile   25% Quantile   50% Quantile   75% Quantile

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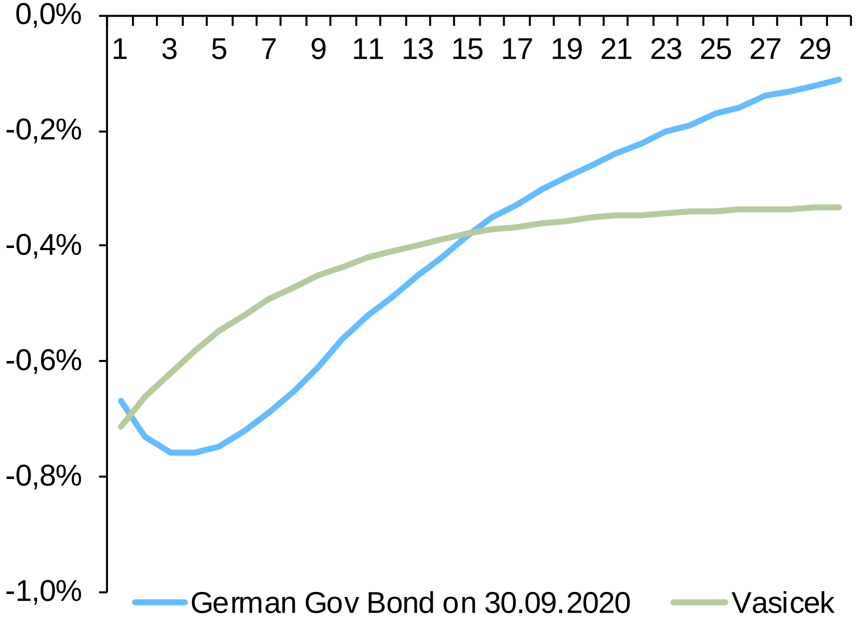
Fitting the initial yield curve

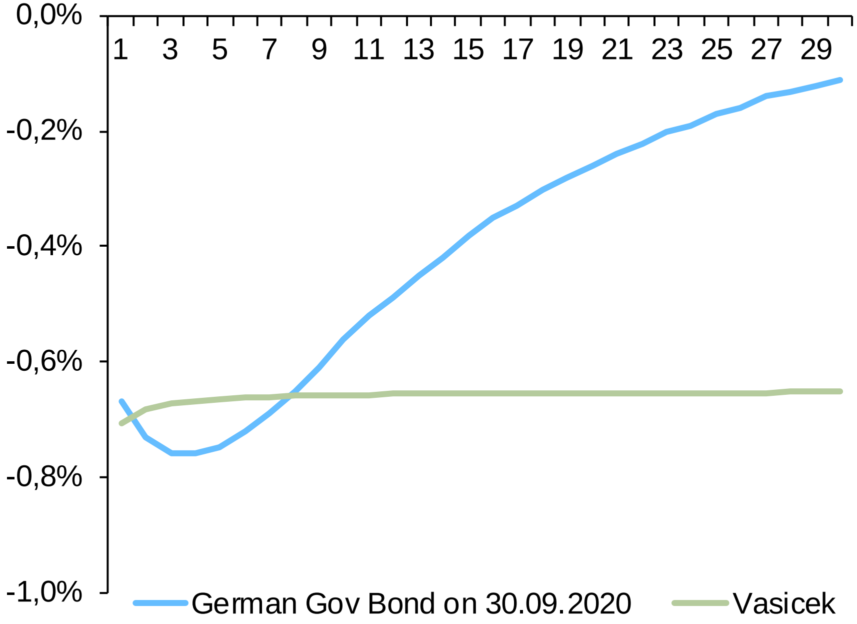
Connection between short rate and yield curve in an affine term-structure model

$$P(t,T) = e^{-R(t,T)\cdot(T-t)} = A(t,T)\cdot e^{-B(t,T)\cdot r(t)}$$

Calibration period 09/1980 – 09/2020



Calibration period 09/2014 – 09/2020



Three parameters (k, θ, σ) are not enough to fit the intial yield curve

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Hull-White (1994) model

Short rate dynamics

$$dr(t) = \big(\theta(t) - a \cdot r(t)\big)dt + \sigma dW(t)$$

Alternative represantation

$$dx(t) = -a \cdot x(t)dt + \sigma dW(t) \qquad r(t) = x(t) + \phi(t)$$

Calibration

long term mean

Calibration using market forward rates f^M

$$\phi(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

Market forward rates e.g. from Nelson-Siegel-Svenson curve of German Government Bonds

➡ Theoretical yield curve fits initial market yield curve

(cf. Brigo and Mercurio, 2006)

reversion and volatility

On one date: via *price of interest rate derivatives*

Minimize the difference between theoretical and empirical price PD (cf. Brigo and Mercurio, 2006)

$$(a,\sigma) = \arg\min_{a,\sigma} \left|PD^{model}(a,\sigma) - PD^{observed}\right|$$

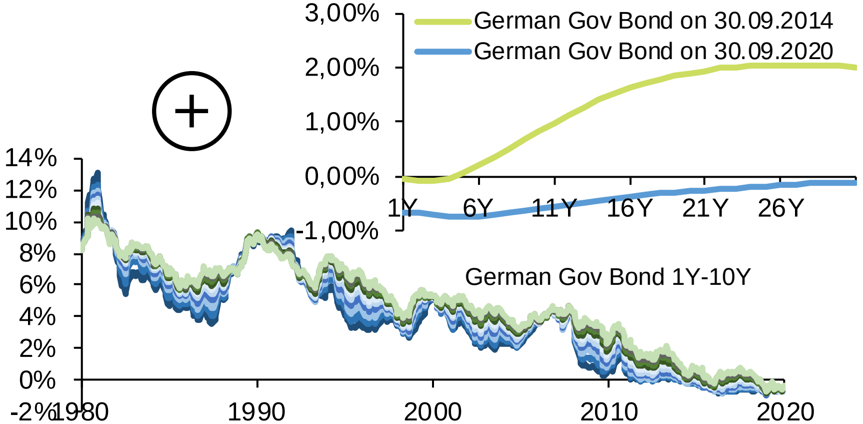
On history of dates: via *spot rate changes* (cf. Aas et. al., 2018)

Minimize the sum of squared differences between theoretical and empirical volatilities of monthly absolute spot rate changes ΔR with maturities $T_k, k = 1 \dots p$.

$$(a,\sigma) = \arg\min_{a,\sigma} \sum_{k=1}^{p} \left(\sigma_{\Delta R}^{model}(T_k; a,\sigma) - \sigma_{\Delta R}^{observed}(T_k)\right)^2$$

Hull-White results

Historical data (German Gov Bonds)



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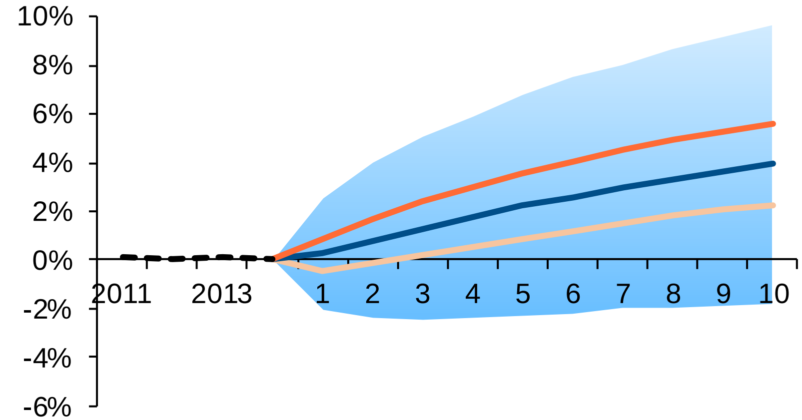

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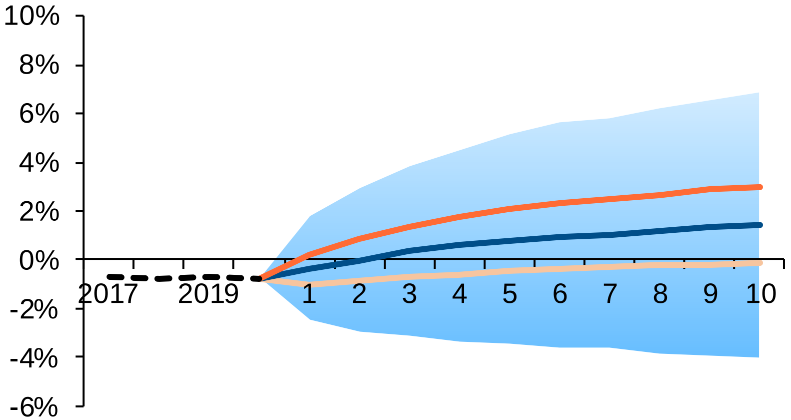

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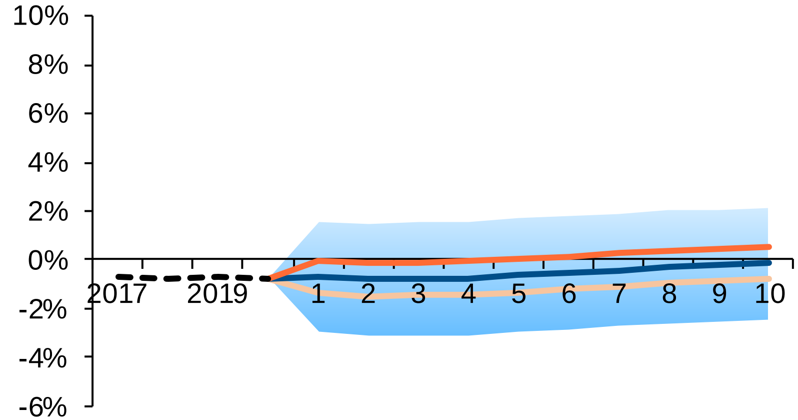
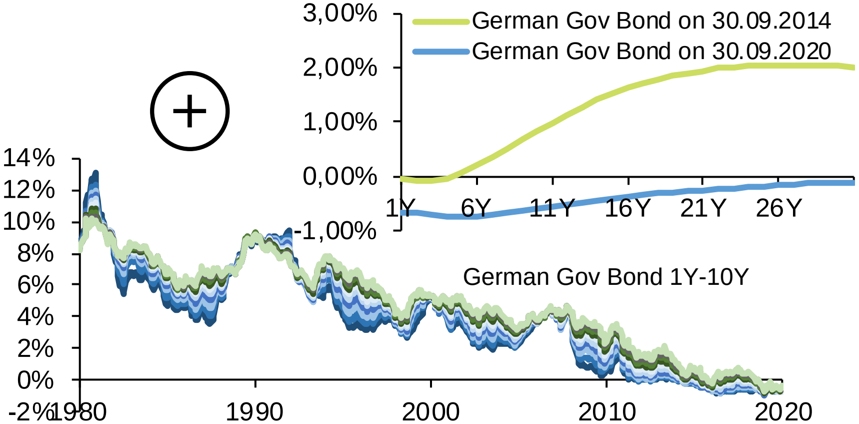


1%-99% Quantile　25% Quantile　50% Quantile　75% Quantile

Hull-White results

Historical data (German Gov Bonds)



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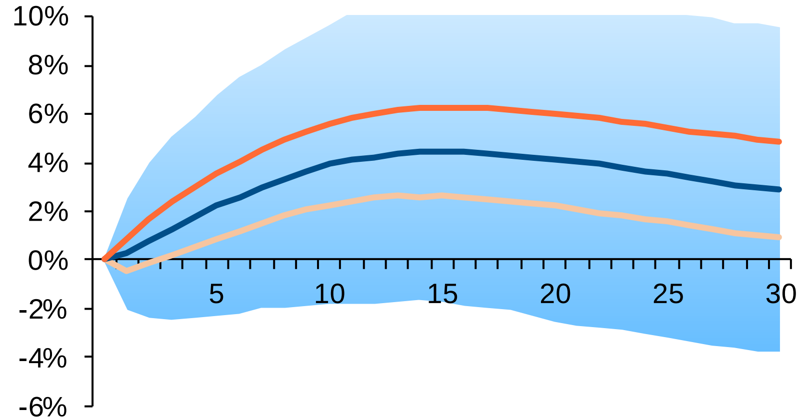

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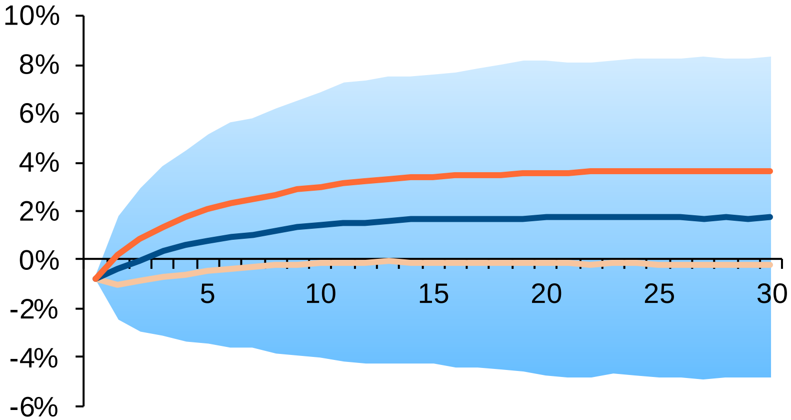

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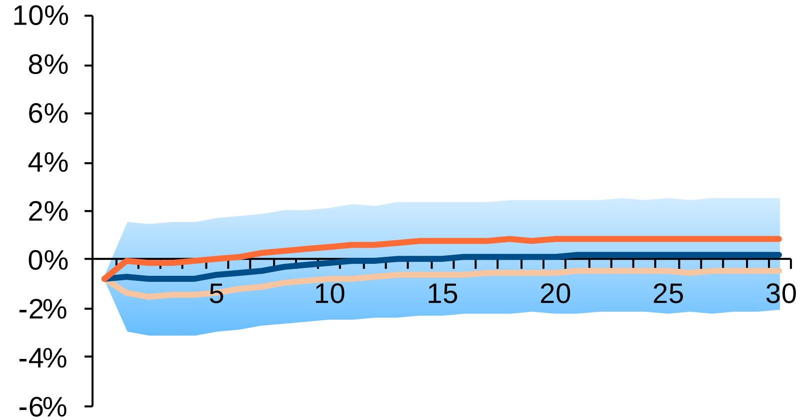

1%-99% Quantile 25% Quantile 50% Quantile 75% Quantile

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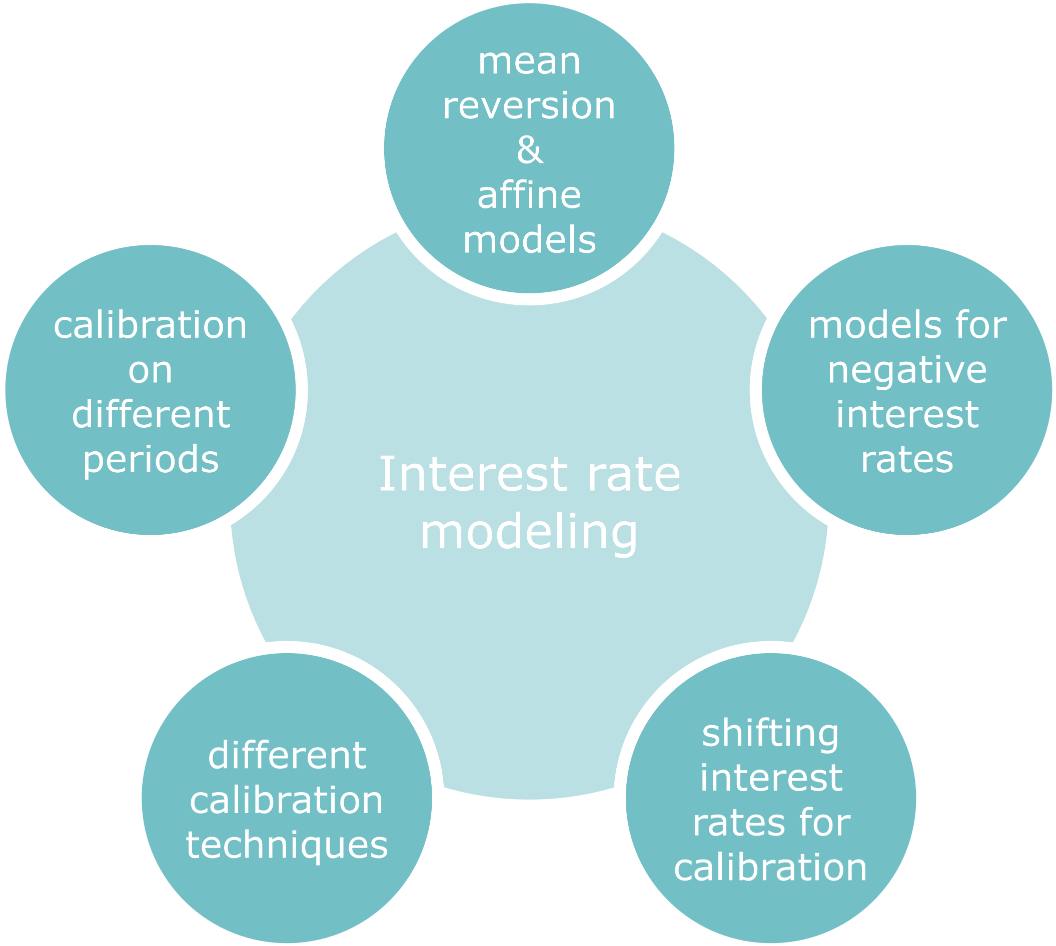
Main points discussed



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Literature

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Orlando, G., Mininni, R. M. and Bufalo, M. (2019) *Interest rates calibration with a CIR model.* The Journal of Risk Finance 20.4, 370-387.

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Thank you for your attention

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