# Creating risk indicators

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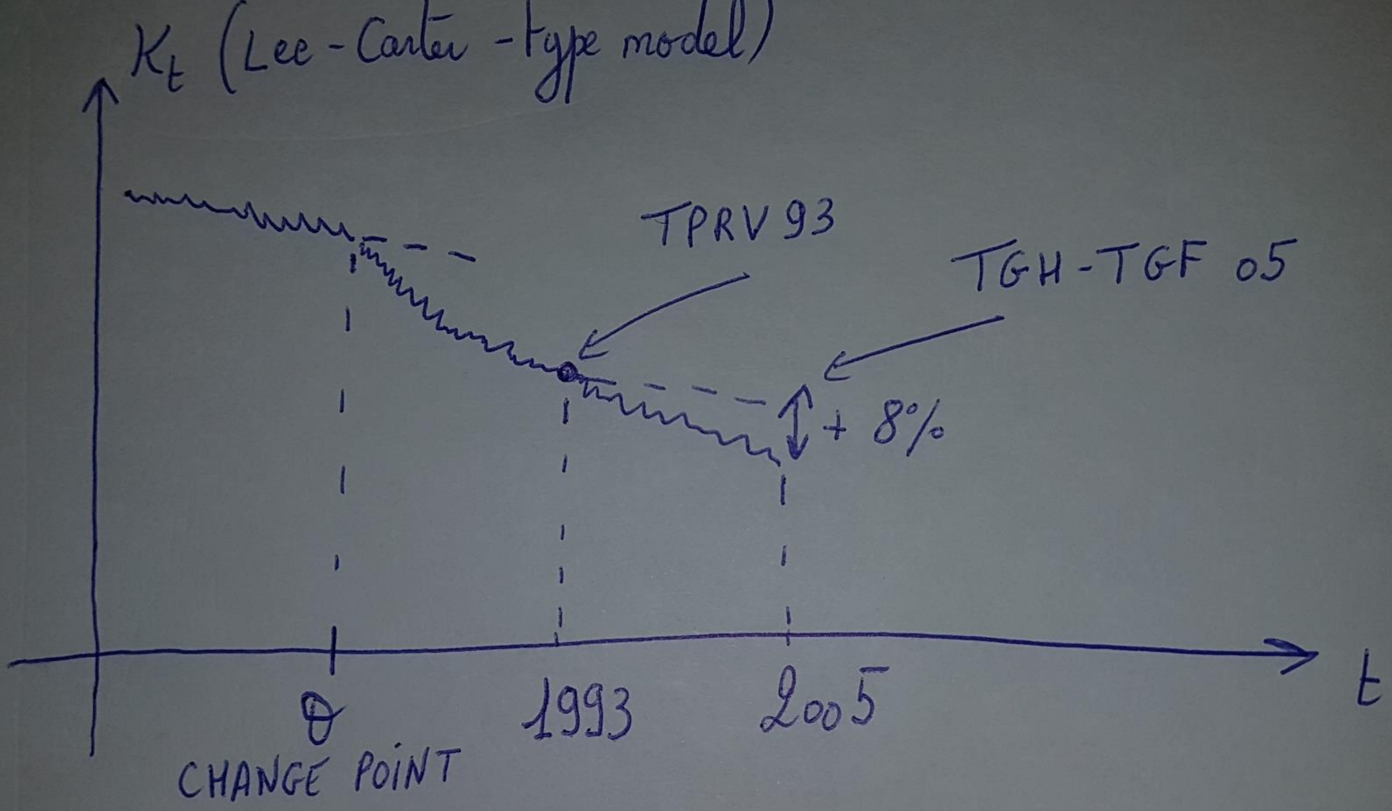
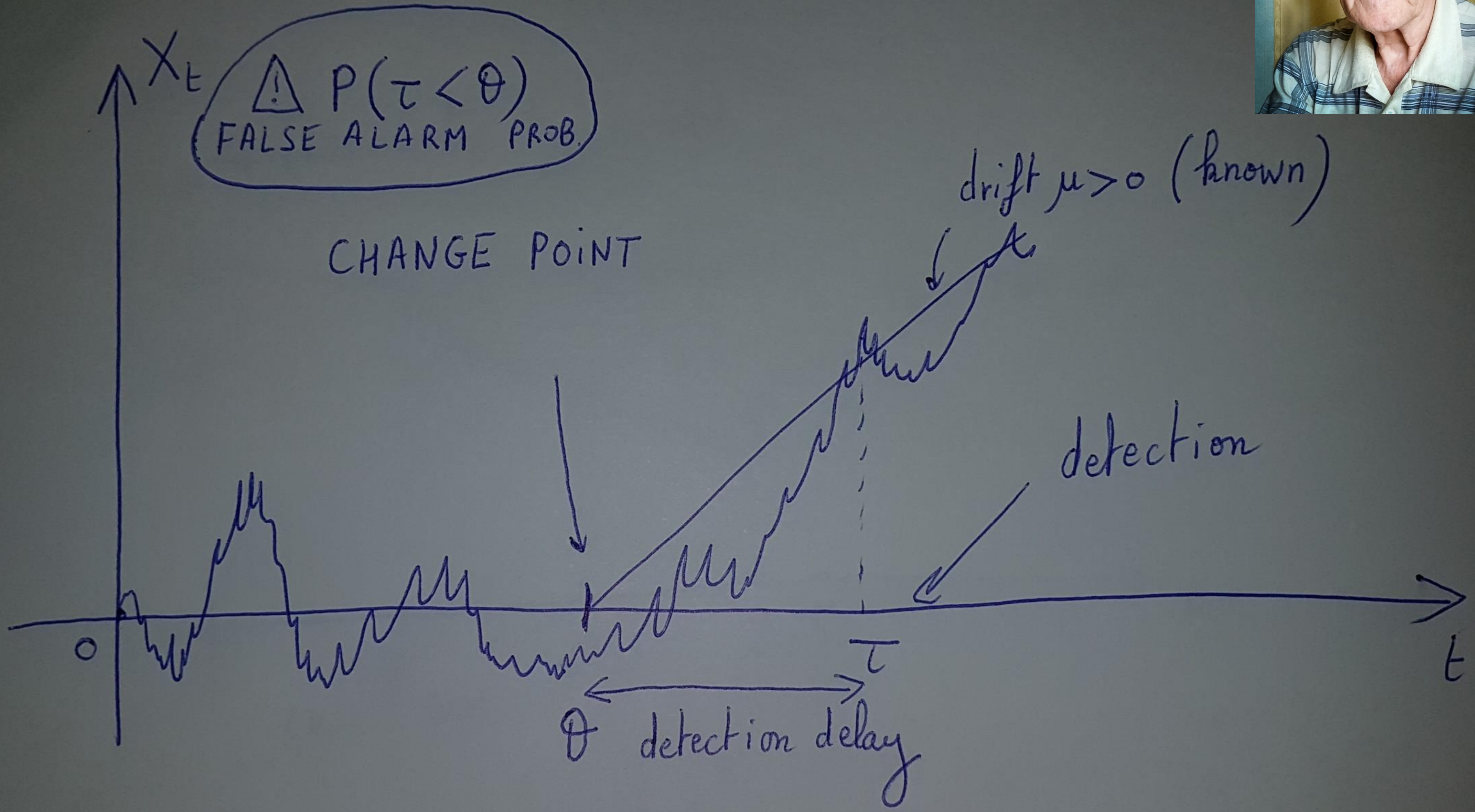
Joint work with N. El Karoui and Y. Salhi


$K_t$ (Lee-Carter-type model)
TPRV 93
TGH-TGF 05
+ 8%
t
θ
CHANGE POINT
1993
2005



$X_t$
⚠ $P(\tau < \theta)$
FALSE ALARM PROB.
CHANGE POINT
drift $\mu > 0$ (known)
detection
0
$\tau$
$t$
$\theta$
detection delay


# Bayesian setup for random change-point

## Brownian framework with abrupt change in the drift

- Based on the conditional distribution of the time of change,
- Formulated as an optimal stopping problem
- Page(1954), Shiryaev(1963), Roberts(1966), Beibel(1988), Moustakides (2004), and many others...

## Poisson framework with abrupt change in intensity

- Based on the conditional distribution of the time of change, with exponential or geometric prior distribution
- More recent studies : Gal (1971), Gapeev (2005), Bayraktar (2005, 2006), Dayanik (2006) for compound Poisson, Peskir, Shyriaev(2009) and others

# Mathematical settings

We consider a portfolio of insured population:

- Let $N = (N_t)_{t \geq 0}$ be a counting process indicating the deaths of policyholders and $\lambda = (\lambda_t)_{t \geq 0}$ its intensity.
- The counting process $N_t$, is available sequentially through the filtration $\mathcal{F}_t = \sigma\{N_s, 0 < s \leq t\}$.
- We suppose that the insurance company relies on a Cox-like model to project her own experienced mortality:

$$\lambda_t = \underline{\rho}\lambda_t^0,$$

  - $\lambda_t^0$ is a reference intensity and $\underline{\rho}$ is a positive parameter.
  - $\lambda^0$ is considered deterministic and may refer whether to a projection of national population/best estimate...

Model risk/parameter uncertainty: **Change-point**

$$\lambda_t = \mathbf{1}_{\{t < \boldsymbol{\theta}\}}\underline{\rho}\lambda_t^0 + \mathbf{1}_{\{t \geq \boldsymbol{\theta}\}}\overline{\rho}\lambda_t^0.$$

Without loss of generality we can assume that $\underline{\rho} = 1$ and let $\boldsymbol{\rho} = \overline{\rho} > 1$.

## Probabilistic formulation

Let $\mathbb{P}_\theta$ (resp. $\mathbb{E}_\theta[\cdot]$) be the probability measure (resp. expectation) induced when the change takes place at time $\theta$

### Example

- For $\theta = 0$, the process is *out-of-control*
- For $\theta = \infty$, the process is *in-control*

**Detect the change-point $\theta$ as quick as possible while avoiding false alarms**

### Optimality Criteria, Lorden (1971)-like

- The detection delay $\mathbb{E}_\theta\left[(N_\tau - N_\theta)^+ \middle| \mathcal{F}_\theta\right]$
- The frequency of false alarm $\mathbb{E}_\infty[N_\tau]$

# Optimization Problem

## Optimization Problem

$$\text{Find } \tau^* \text{ such that } C(\tau^*) = \inf_\tau \mathbf{sup}_{\theta\in[0,\infty]} \mathbf{ess\,sup} \mathbb{E}_\theta \left[ (N_\tau - N_\theta)^+ \middle| \mathcal{F}_\theta \right]$$
$$\text{subject to } \mathbb{E}_\infty[N_\tau] = \omega.$$

## Assumption

1. $\int_0^t \lambda_s ds < \infty, \quad \mathbb{P}_\infty, \mathbb{P}_0$-a.s.
2. $N_\infty = \infty \qquad \mathbb{P}_\infty, \mathbb{P}_0$-a.s.

# Optimality of the Cusum procedure (1/7)

Let the Radon-Nikodym density of $\mathbb{P}_0$ with respect to $\mathbb{P}_\infty$ be defined as

$$\left.\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}\right|_{\mathcal{F}_t} = \exp U_t,$$

where $U_t = \log(\rho)N_t + (1-\rho)\int_0^t \lambda_s^0 ds$ is the log-likelihood ratio.
Let $V(x)$ be the CUSUM process; with head-start $0 \leq x < m$; defined as

$$V_t(x) = U_t - (-x) \wedge \underline{U}_t \tag{1}$$

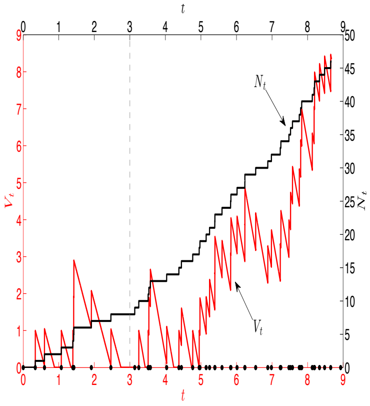
where $\underline{U}_t$ is the running infimum of $U$, i.e. $\underline{U}_t = \inf_{s\leq t} U_s$.
The process $V(x)$ measures the size of the drawup, comparing the present value of the process $U$ to its historical infimum $\underline{U}$.
Let $\tau_m(x)$ be the fist hitting time of $V(x)$ of the barrier $m$, i.e.

$$\tau_m(x) = \inf\{t \geq 0, V_t(x) \geq m\}.$$

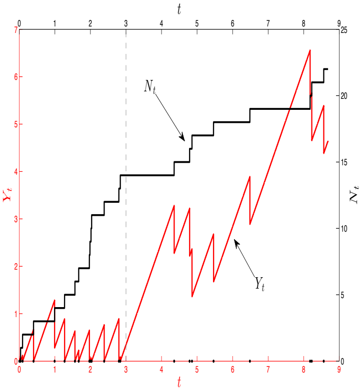
**Theorem**

*If* $\mathbb{E}_\infty[N_{\tau_m(0)}] = \omega$ *then* $\tau_m(0)$ *is optimal, i.e.* $\inf_\tau C(\tau) = C(\tau_m(0))$

# Typical paths with change of regime at date 3

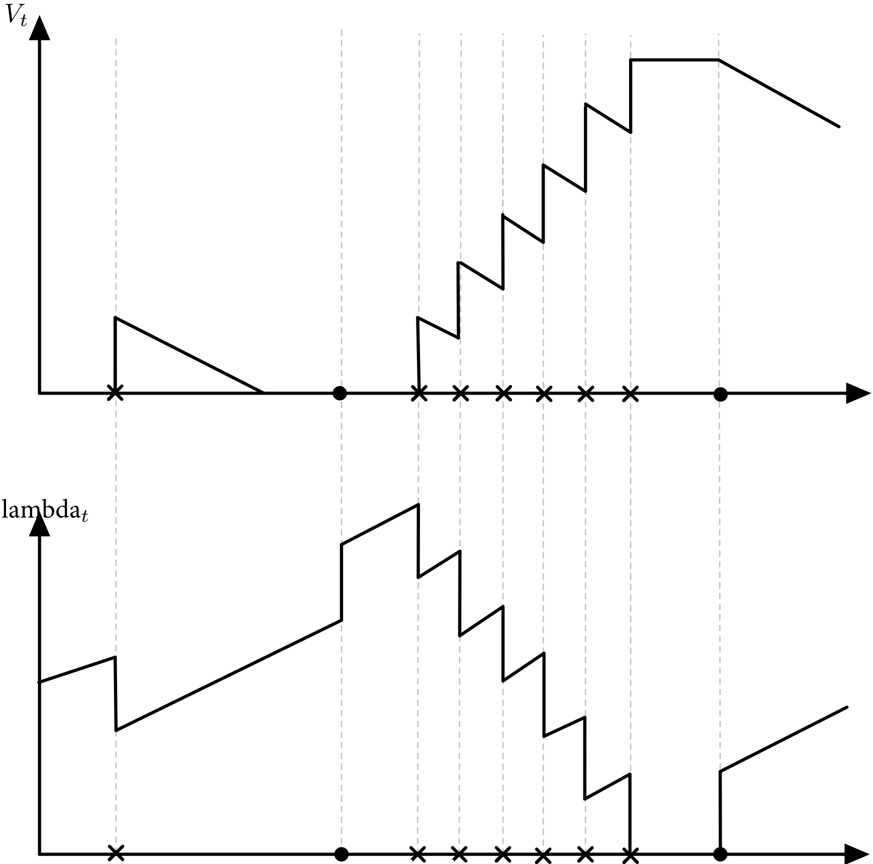



(a) Processes $N$ and $V_t$




(b) Processes $N$ and $Y_t$

Figure: Sample paths, for $\rho = 1.5$, of the cusum processes $N$, $V^\rho$ (left) and $N$, $Y_t^\rho$ for $\rho = 0.5$ (right ).


$V_t$
lambda$_t$

Why is quick detection important in insurance?
How to choose parameter rho?

| | females | | | | males | | | |
|---|---|---|---|---|---|---|---|---|
| | Doubled improvements | | Mortality level at 80% of the expected | | Doubled improvements | | Mortality level at 80% of the expected | |
| | pension value | interest rate | pension value | interest rate | pension value | interest rate | pension value | interest rate |
| **55** | +5.4% | +32bp | +3.1% | +19bp | +6.7% | +42bp | +3.7% | +24bp |
| **65** | +5.76% | +43bp | +4.7% | +36bp | +7% | +57bp | +5.7% | +48bp |
| **75** | +5.2% | +55bp | +7.6% | +80bp | +6.3% | +74bp | +9.1% | +107bp |
| **85** | +3.6% | +60bp | +13.2% | +207bp | +4.3% | +84bp | +15.4% | +281bp |

TABLE: TGH05/TGF05 with flat interest rate of 3%

# Monitoring Mortality

## Sounding an alarm for the change $\rho^{\text{Hyp}} \to \rho^{\text{Targer}}$

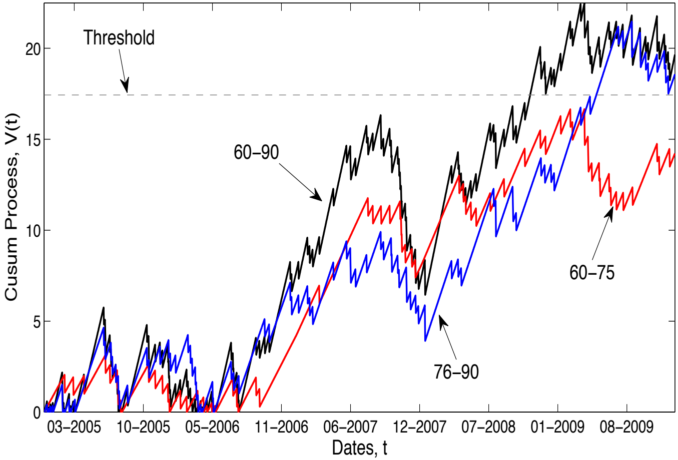
- We simulate deaths on the portfolio with different levels $\rho^{\text{Target}} = 95\%$, $90\%$ and $85\%$ s.t.

$$D(x,t) \sim \text{Pois}(\rho^{\text{Target}} \times L(x,t) \times \mu^{\text{ERM00}}(x,t))$$

- We suppose that *the actuary* made an assumption of $\rho^{\text{Hyp}} = 100\%$
- We set-up the monitoring/surveillance on the observed deaths and try to detect a change from $\rho^{\text{Hyp}} = 100\%$ to $\rho^{\text{Target}} = 95\%$, $90\%$ and $85\%$ respectively.
- We test different sizes of the portfolio small sized 1000, 5000 and a (relatively) large 10000 and compare the results

# Monitoring Mortality

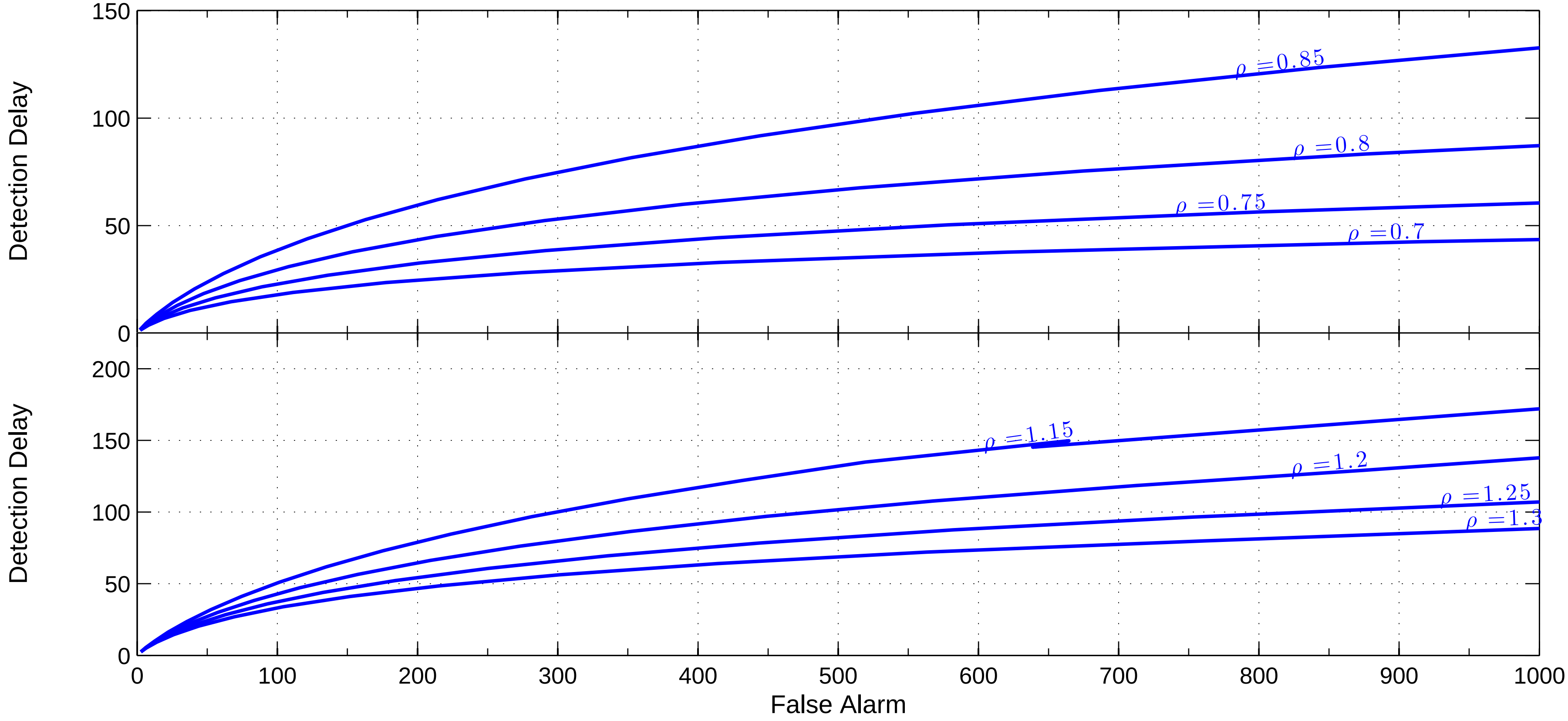
## Sounding an alarm for the change $\rho^{\text{Hyp}} = 100\% \rightarrow \rho^{\text{Target}} = 95\%$



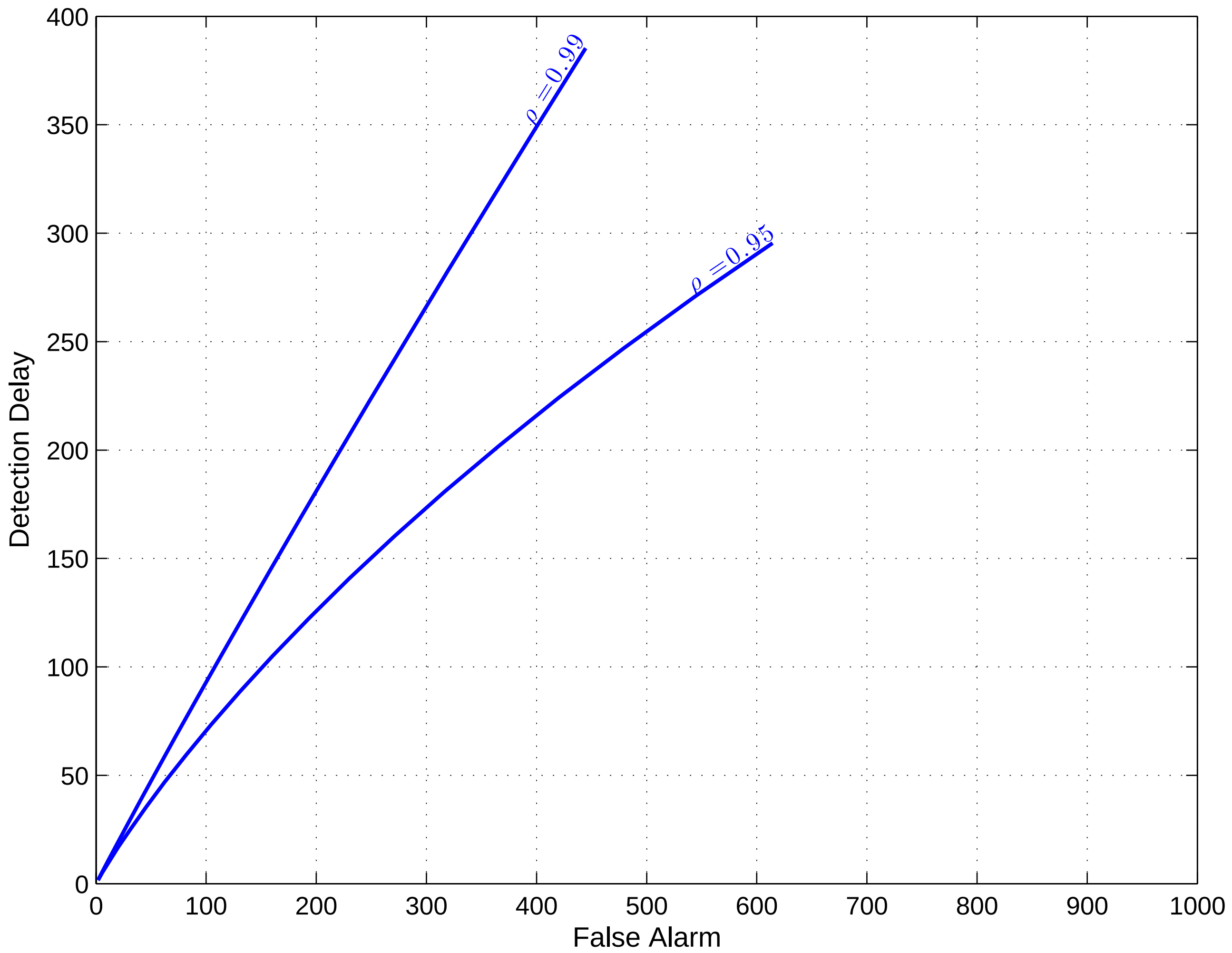
# Detection Delay

## Impact of Portfolio Size and Age Tranches

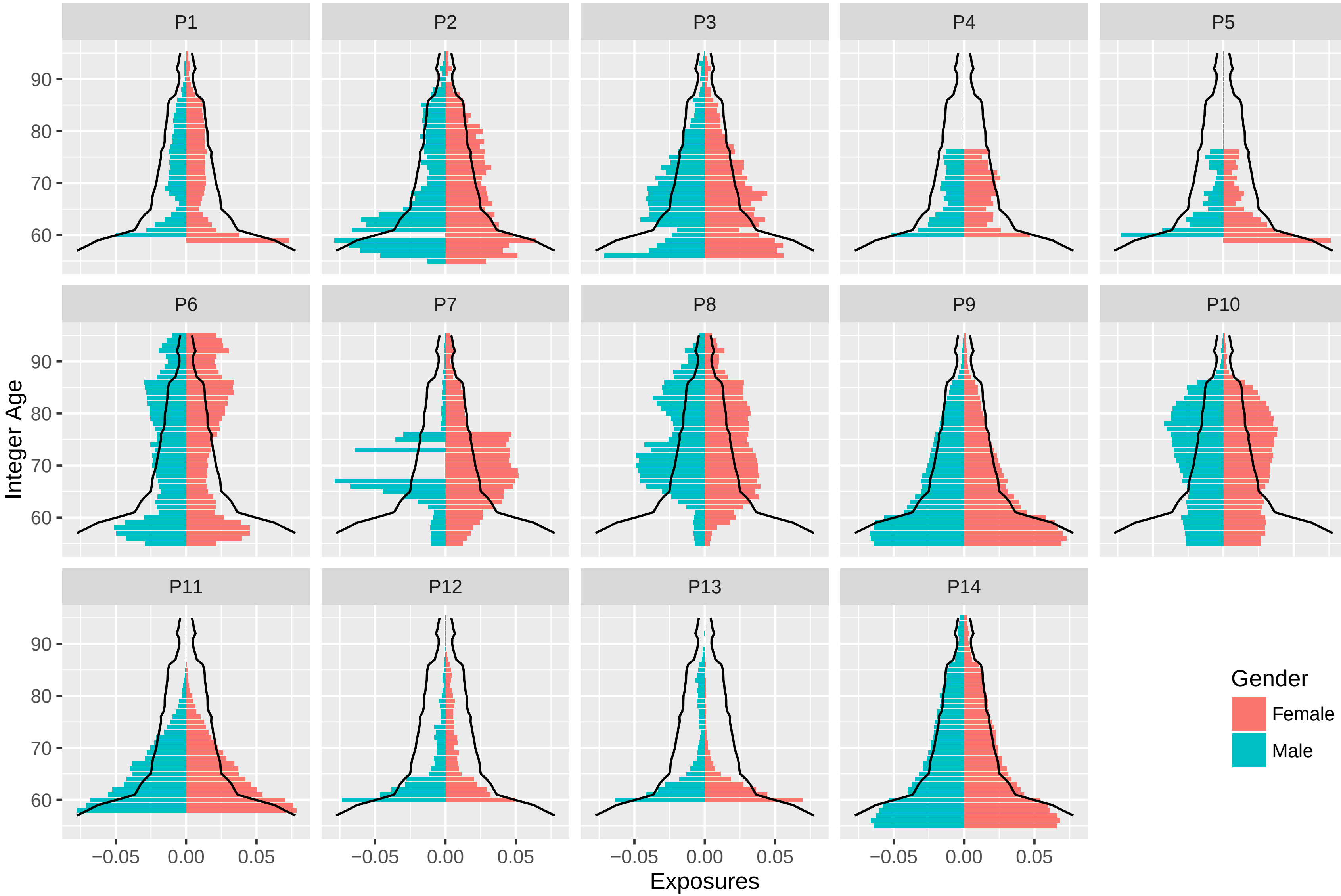
| | Size | 1000 | | | 5000 | | | 10000 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Hyp. \ Ages | 60-90 | 60-75 | 76-90 | 60-90 | 60-75 | 76-90 | 60-90 | 60-75 | 76-90 |
| deaths | 100% → 95% | 596 | 710 | 498 | 246 | 99 | 107 | 240 | 99 | 106 |
| | 100% → 90% | 244 | 320 | 186 | 106 | 55 | 59 | 112 | 55 | 58 |
| | 100% → 85% | 92 | 122 | 100 | 58 | 35 | 36 | 61 | 34 | 36 |
| time | 100% → 95% | 1086 | 1130 | 1120 | 576 | 617 | 422 | 308 | 327 | 212 |
| | 100% → 90% | 931 | 1124 | 947 | 276 | 373 | 241 | 151 | 192 | 127 |
| | 100% → 85% | 707 | 980 | 734 | 161 | 247 | 159 | 84 | 127 | 80 |

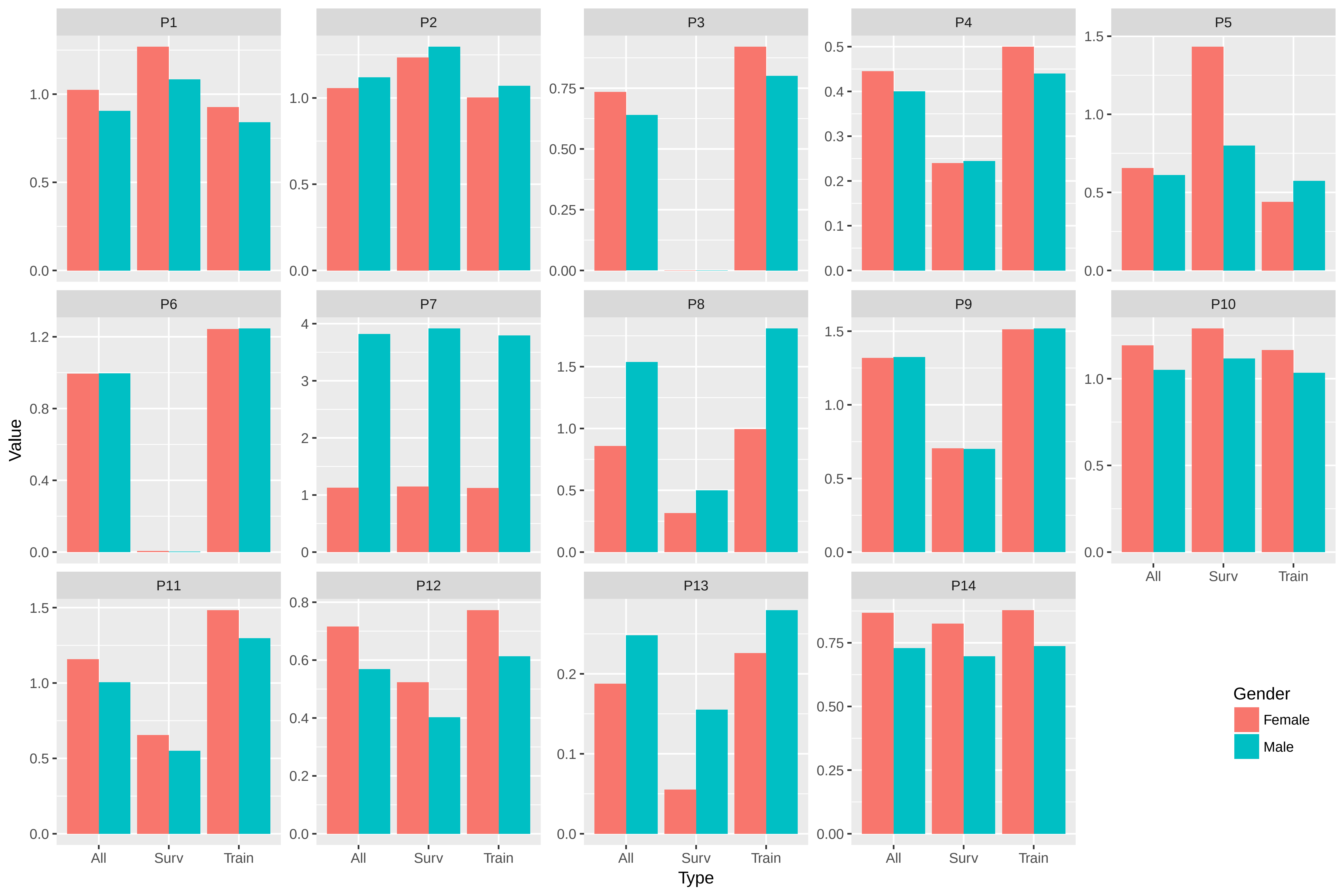


Detection Delay
Detection Delay
False Alarm
$\rho = 0.85$
$\rho = 0.8$
$\rho = 0.75$
$\rho = 0.7$
$\rho = 1.15$
$\rho = 1.2$
$\rho = 1.25$
$\rho = 1.3$
0
50
100
150
200
0
100
200
300
400
500
600
700
800
900
1000

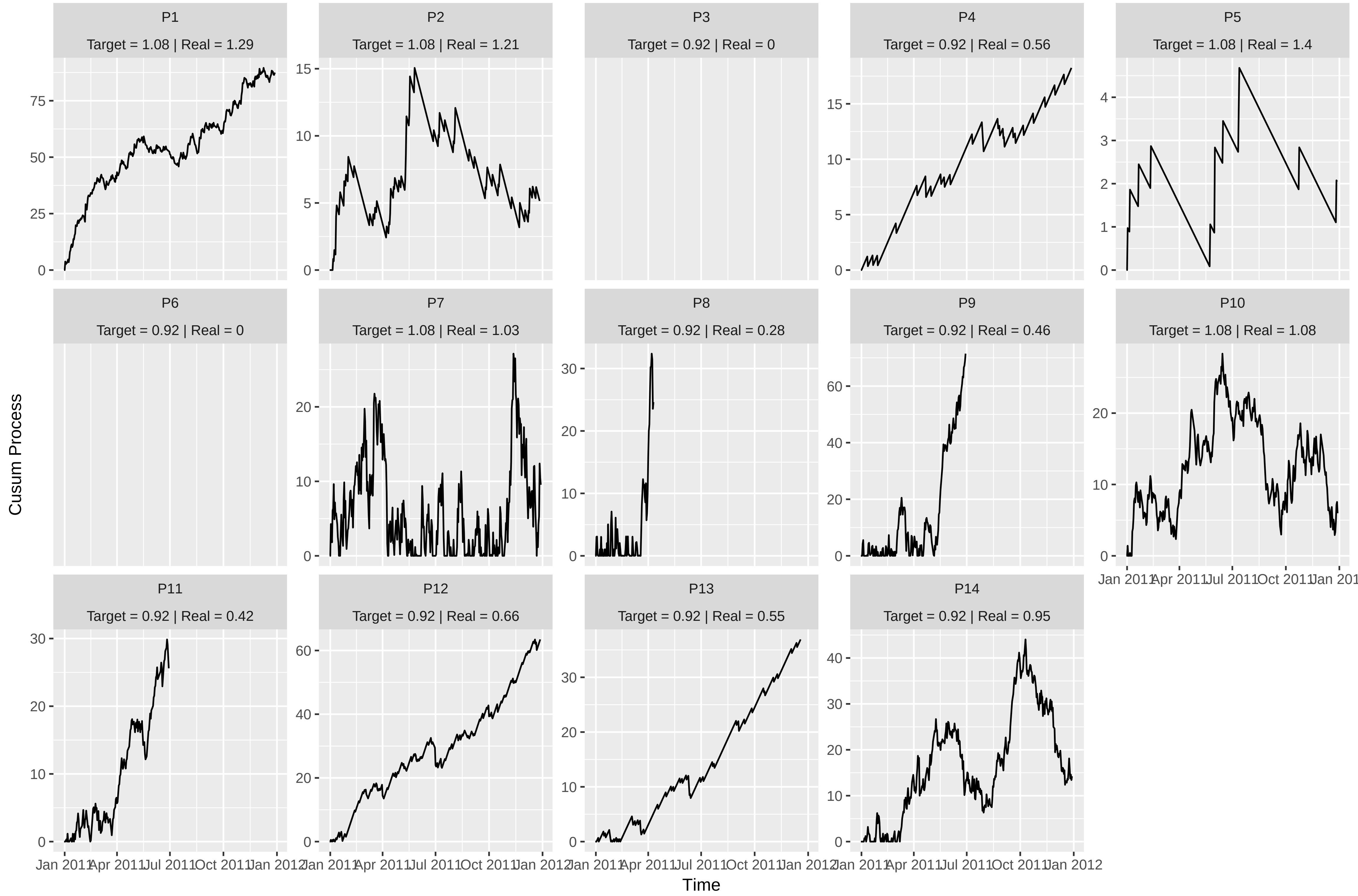


Detection Delay
False Alarm
$\rho = 0.99$
$\rho = 0.95$
0
50
100
150
200
250
300
350
400
0
100
200
300
400
500
600
700
800
900
1000

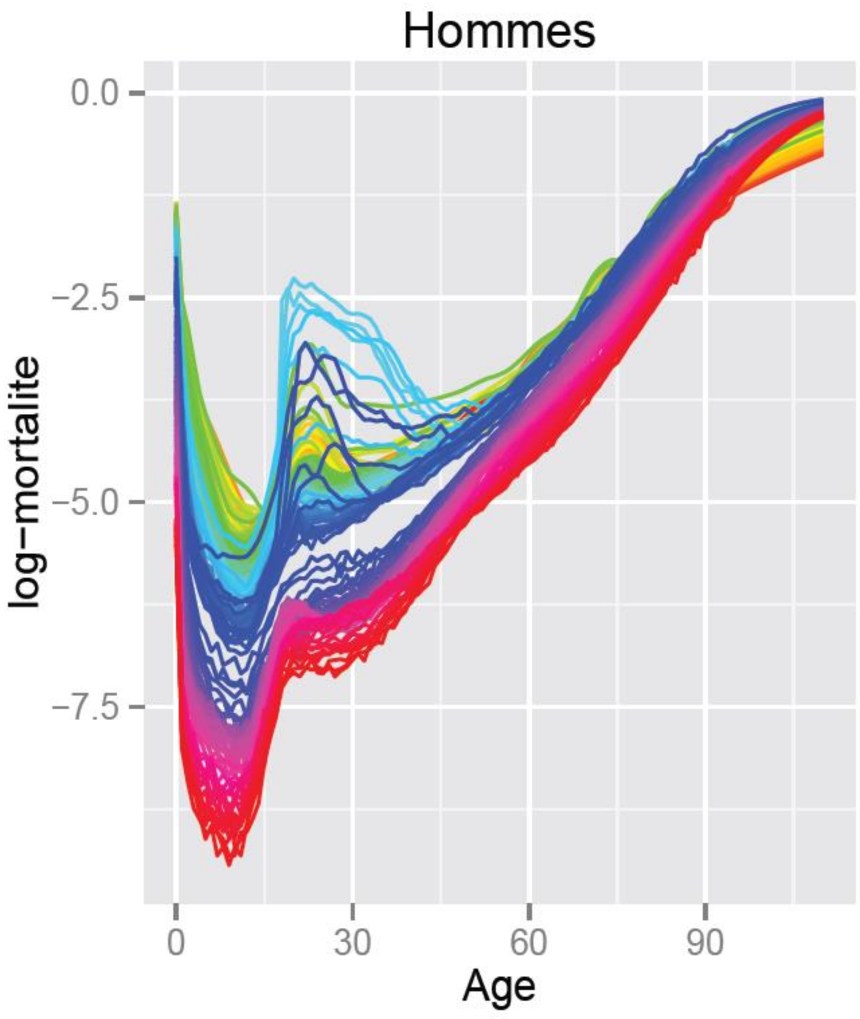

P1
P2
P3
P4
P5
P6
P7
P8
P9
P10
P11
P12
P13
P14
Integer Age
90
80
70
60
−0.05
0.00
0.05
Exposures
Gender
Female
Male



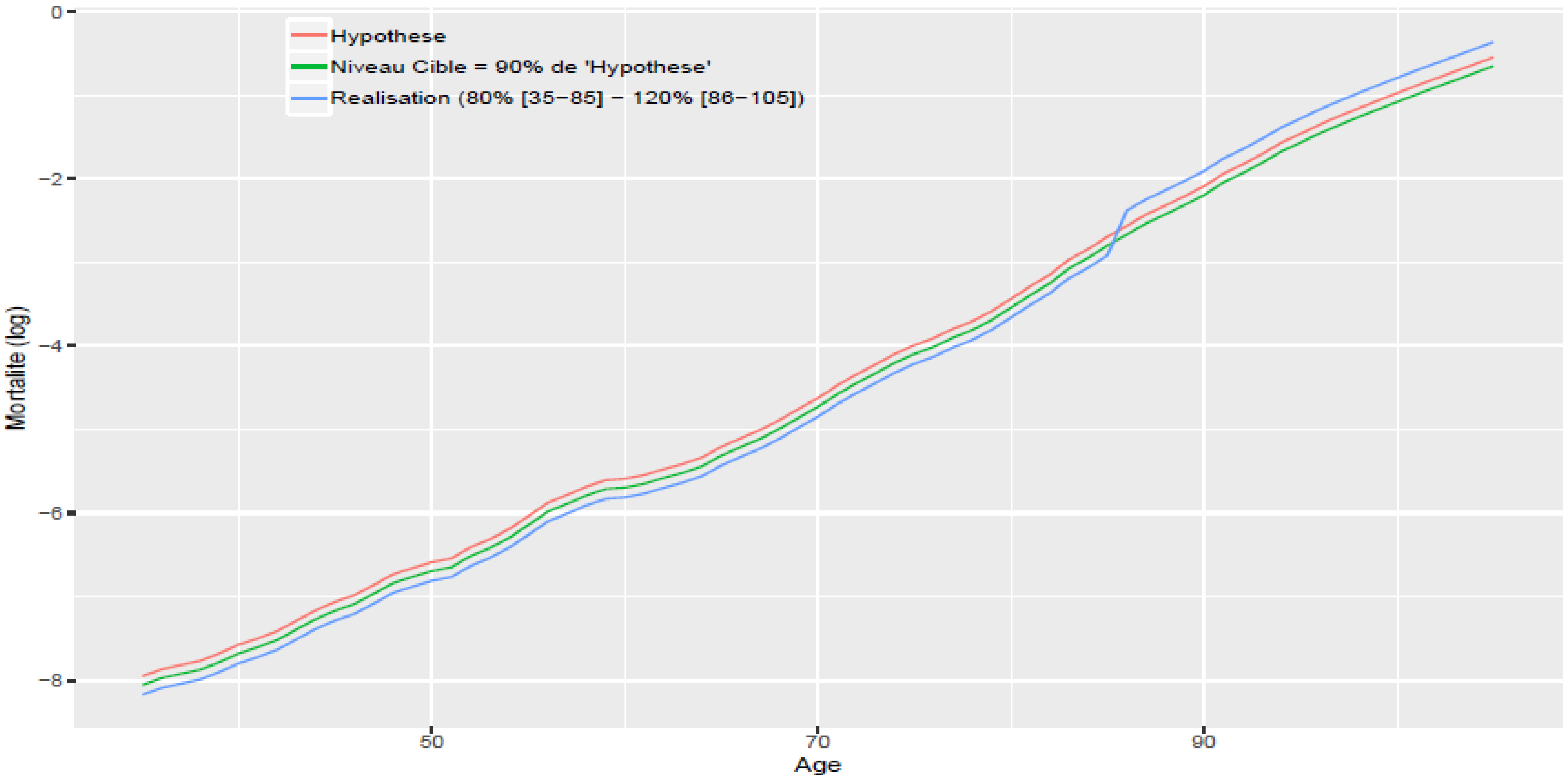
P1
P2
P3
P4
P5
P6
P7
P8
P9
P10
P11
P12
P13
P14
Value
Type
All
Surv
Train
Gender
Female
Male


P1
Target = 1.08 | Real = 1.29
P2
Target = 1.08 | Real = 1.21
P3
Target = 0.92 | Real = 0
P4
Target = 0.92 | Real = 0.56
P5
Target = 1.08 | Real = 1.4
P6
Target = 0.92 | Real = 0
P7
Target = 1.08 | Real = 1.03
P8
Target = 0.92 | Real = 0.28
P9
Target = 0.92 | Real = 0.46
P10
Target = 1.08 | Real = 1.08
P11
Target = 0.92 | Real = 0.42
P12
Target = 0.92 | Real = 0.66
P13
Target = 0.92 | Real = 0.55
P14
Target = 0.92 | Real = 0.95
Cusum Process
Time
Jan 2011
Apr 2011
Jul 2011
Oct 2011
Jan 2012

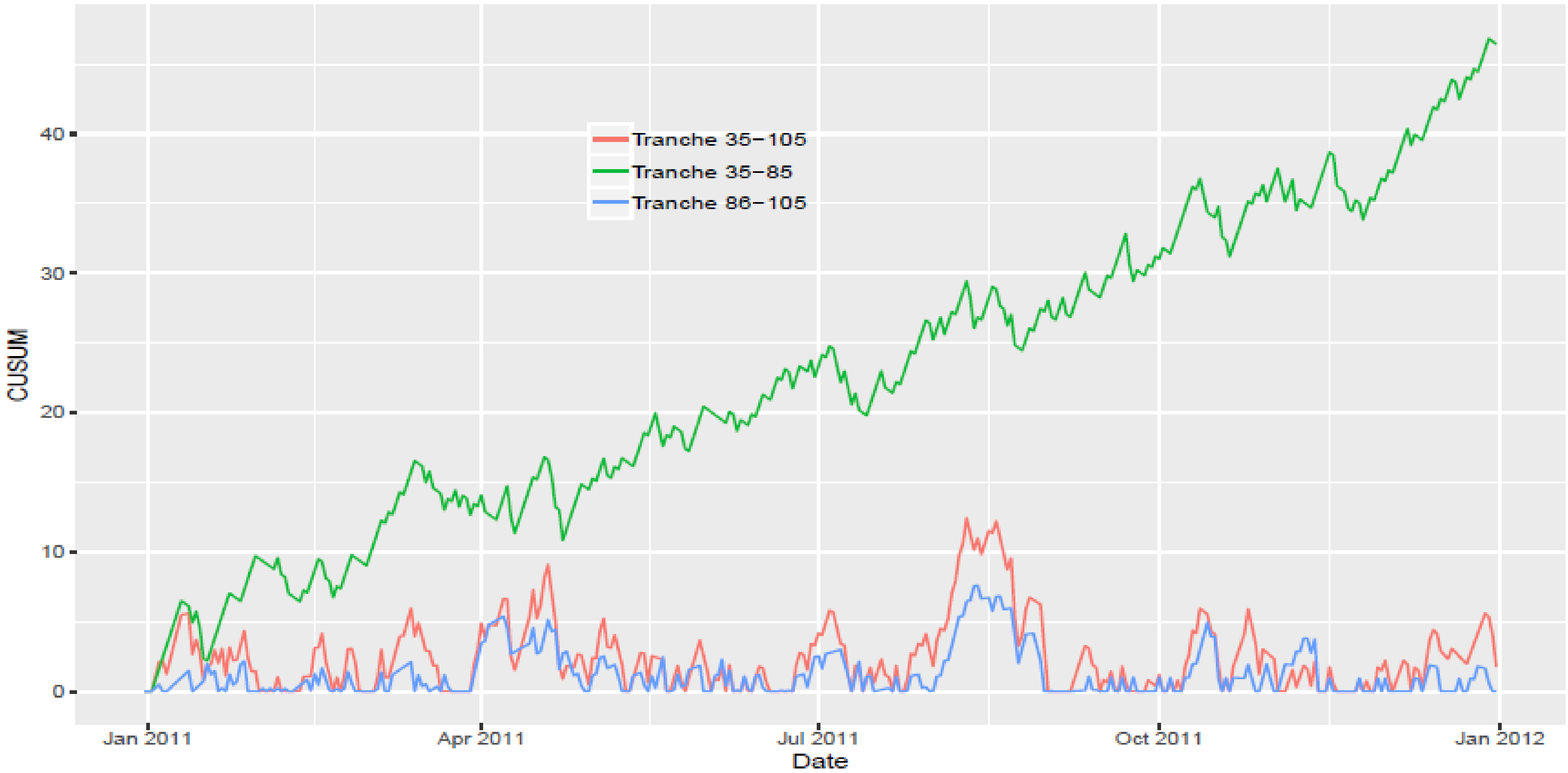
# France, 1816-2000

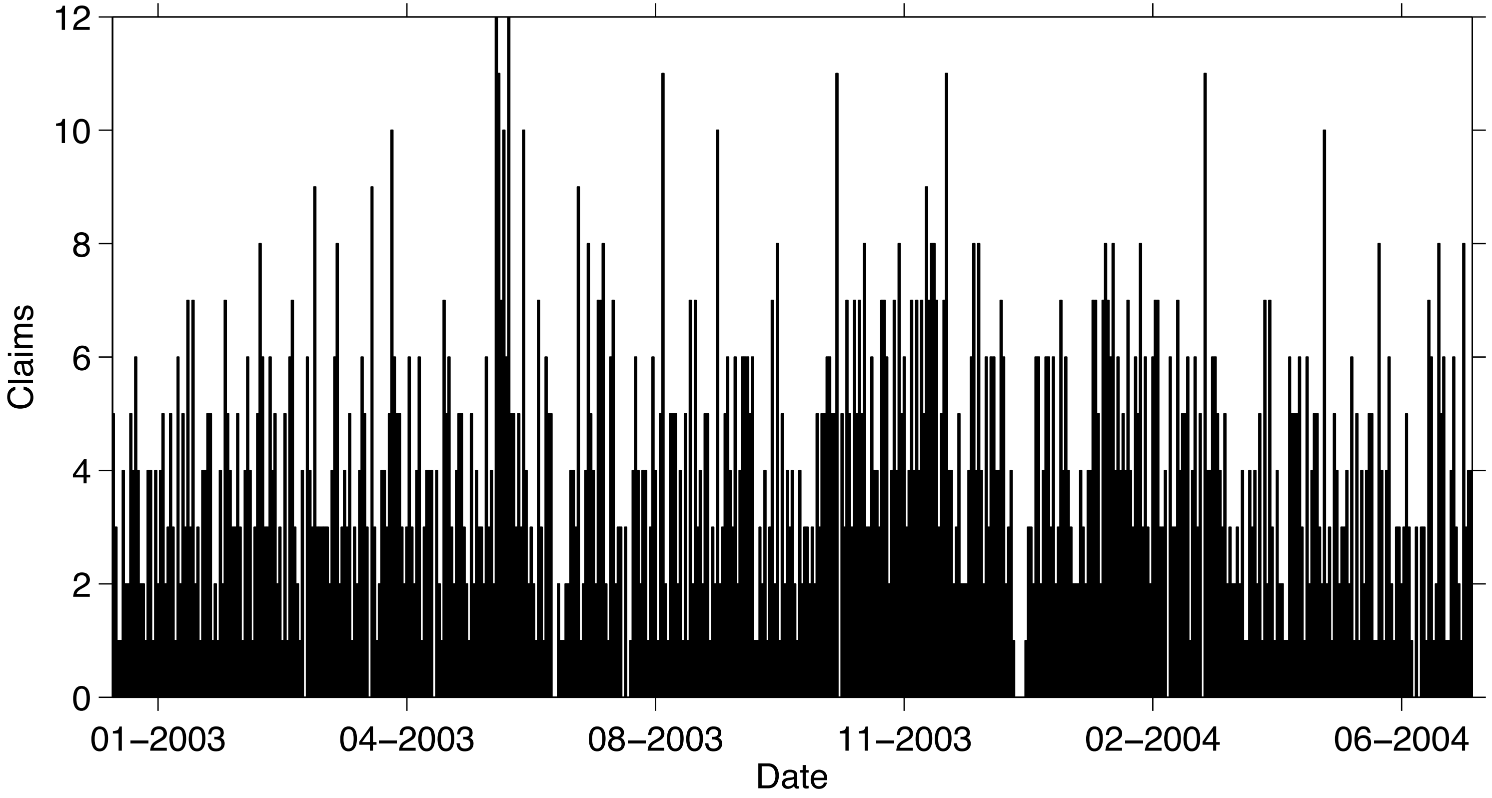


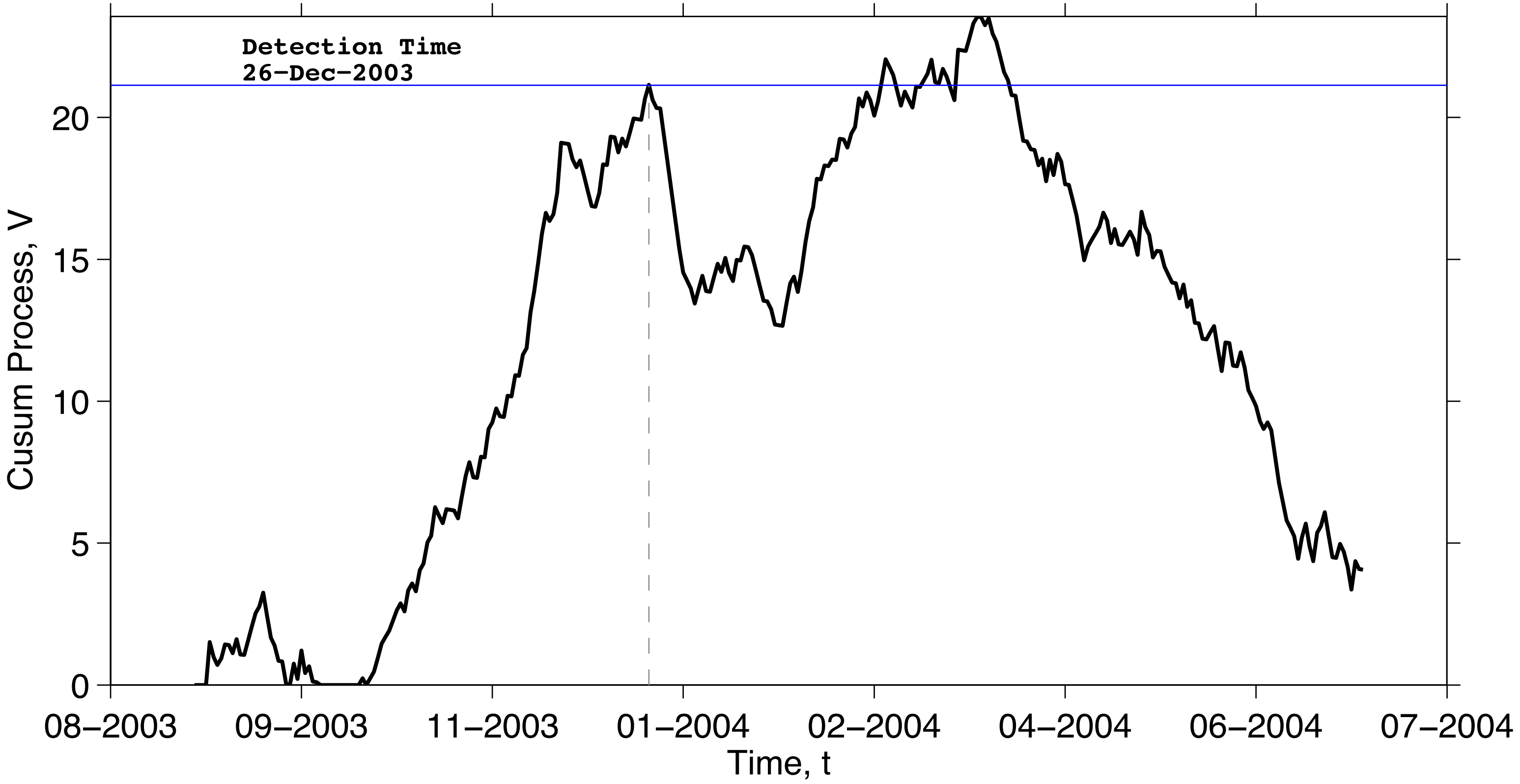
# Impact of rectangularization (without transhumanism)

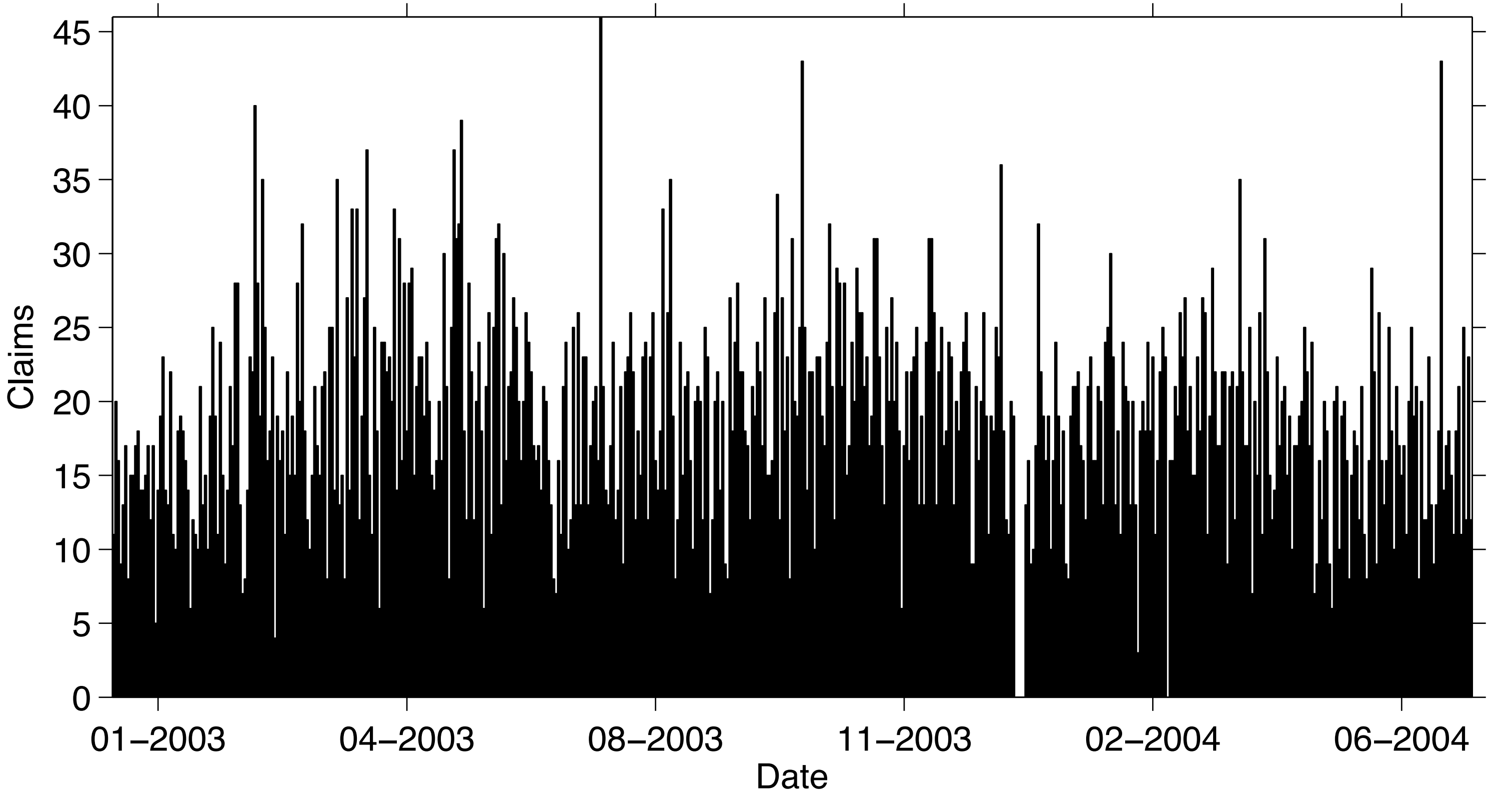


# Pour une fois, Verts >> Rouges et bleus

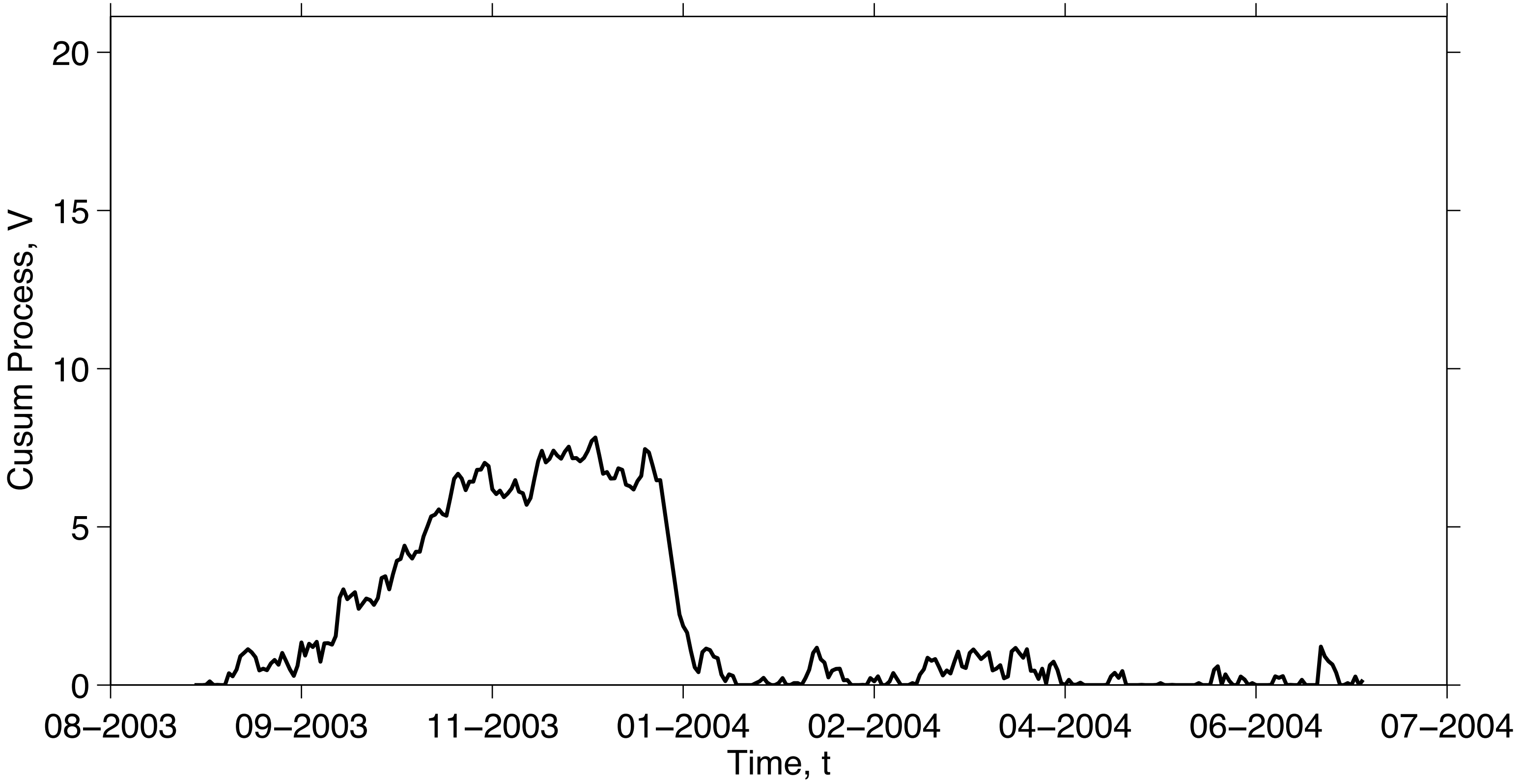



12
10
8
6
4
2
0
Claims
01−2003
04−2003
08−2003
11−2003
02−2004
06−2004
Date


Detection Time
26-Dec-2003
Cusum Process, V
Time, t
0
5
10
15
20
08−2003
09−2003
11−2003
01−2004
02−2004
04−2004
06−2004
07−2004



Claims
Date
0
5
10
15
20
25
30
35
40
45
01−2003
04−2003
08−2003
11−2003
02−2004
06−2004



Cusum Process, V
20
15
10
5
0
08−2003
09−2003
11−2003
01−2004
02−2004
04−2004
06−2004
07−2004
Time, t

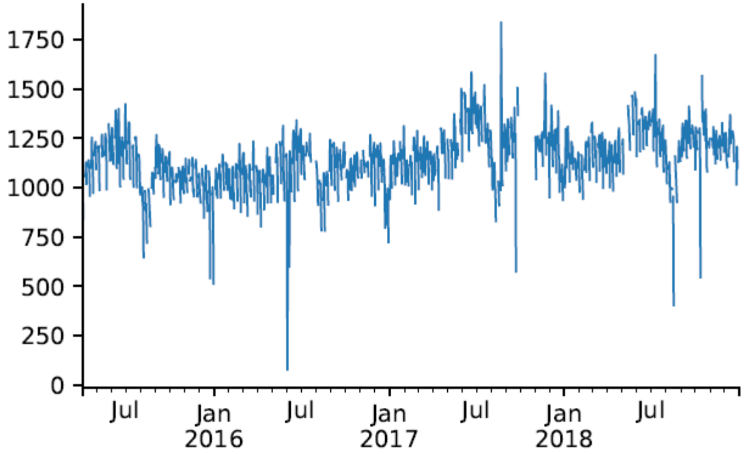
# Business-motivated research question

- Can we detect a surge in calls at 

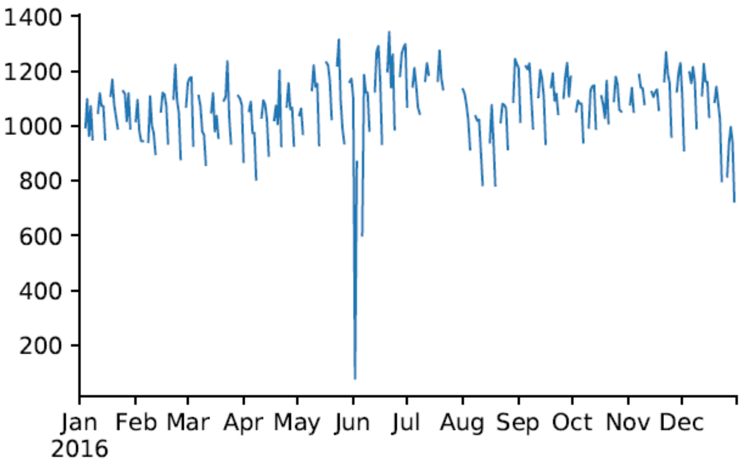
fast enough to onboard extra staff members to handle it?

-> Handle Seasonality

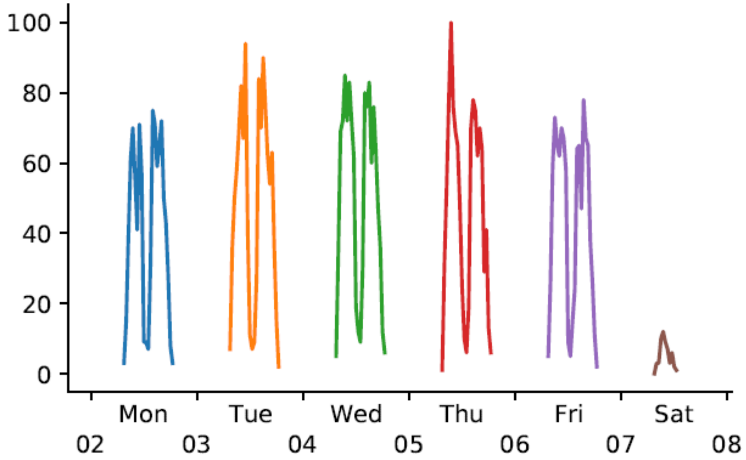
# Seasonality



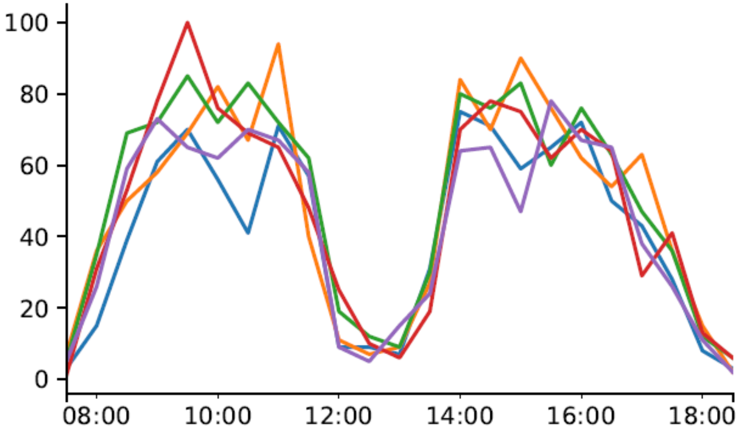

(a) Number of calls each weekday over the period ranging from April 2015 to January 2019.




(b) Number of calls each weekday over the year 2016.




(c) Number of calls during the week ranging from Monday 2nd to Saturday 7th of January 2017.




(d) Number of calls during each day from Monday 2nd to Friday 6th of January 2017.

Figure 1: Seasonality in call arrivals: (a) yearly, (b) monthly, (c) daily and (d) hourly.

# What do we want to detect?

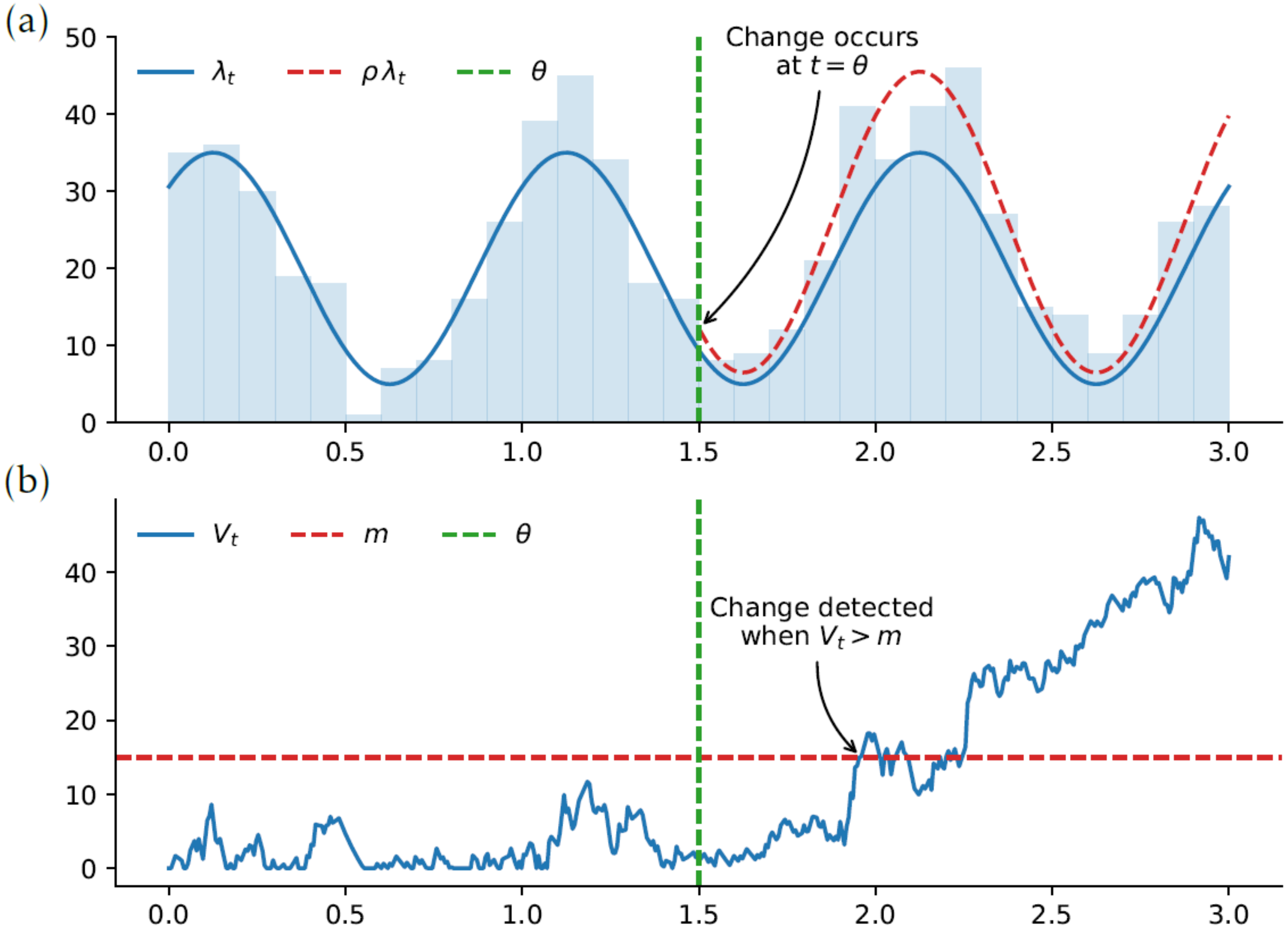



Figure 2: (a) Example of seasonal intensity (solid line) with simulated records and a change-point at time $\theta = 1.5$ with 30% increase of the intensity (dashed line). (b) Sample path of the CUSUM process $V$.

# If we ignore seasonality: false alarms +++

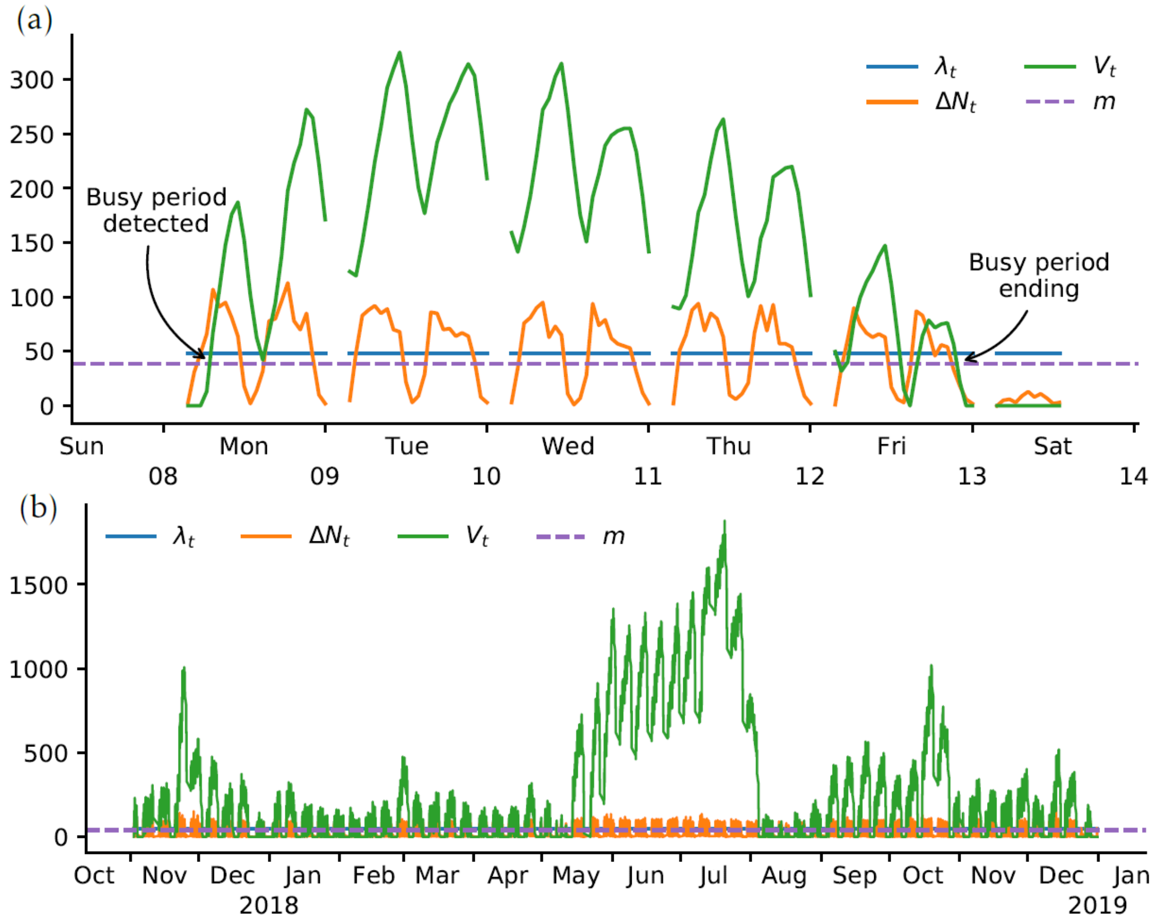



Figure 8: The results of CUSUM with a naive $\lambda_t \equiv \lambda$ over (a) the week starting 2018-01-08 and (b) the whole test period. The number of calls arriving in each half-hour time slot is denoted $\Delta N_t$.

# Taking into account seasonality: it « works »!

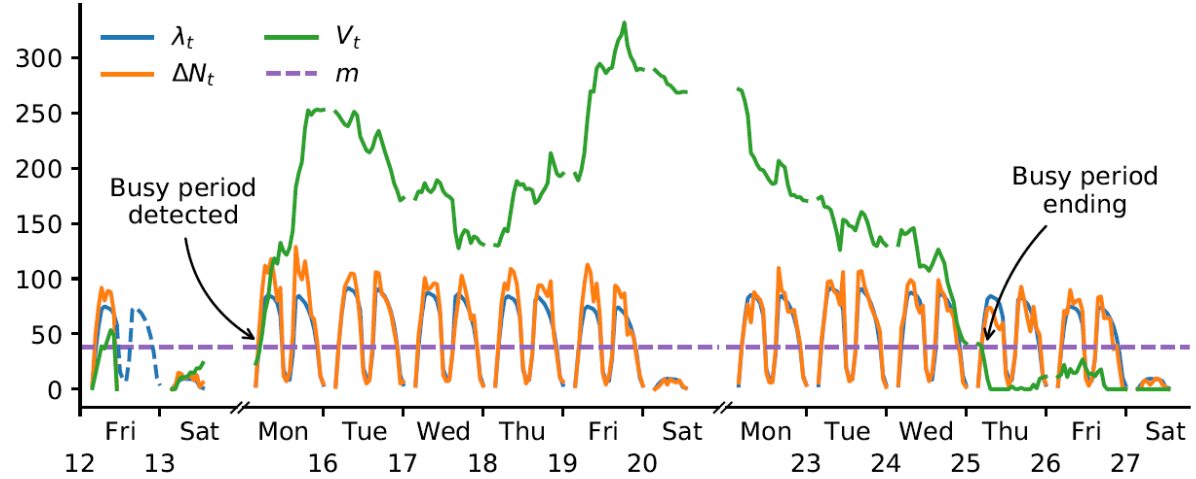



Figure 9: CUSUM algorithm results for a few weeks starting from 2018-10-12. The call center closed at noon on the Friday the 12th, which is unusual. In the CUSUM we modified $\lambda_t$ to be zero for that afternoon, though here we plotted the original $\lambda_t$.

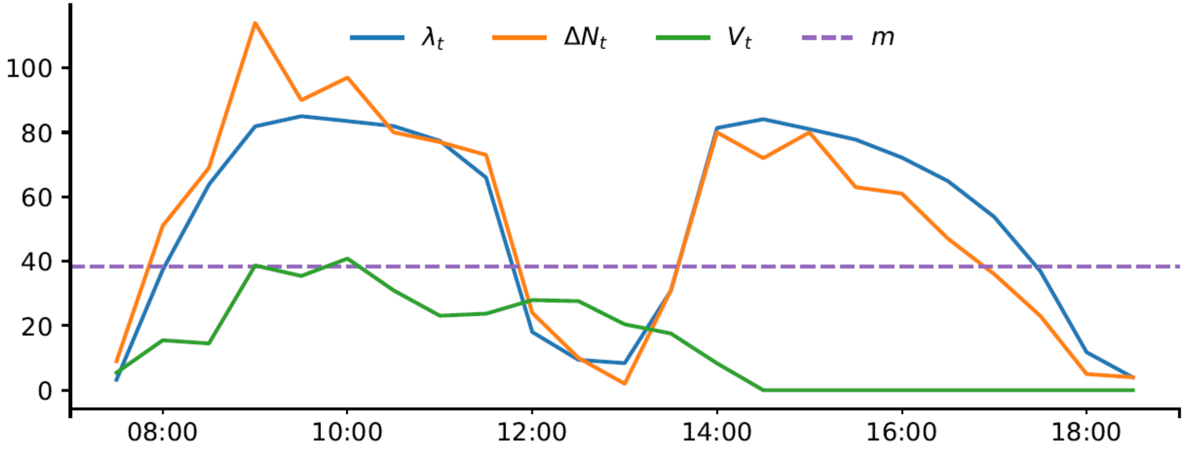



Figure 10: CUSUM algorithm results for the date 2018-06-15.

# What if some calls are postponed?

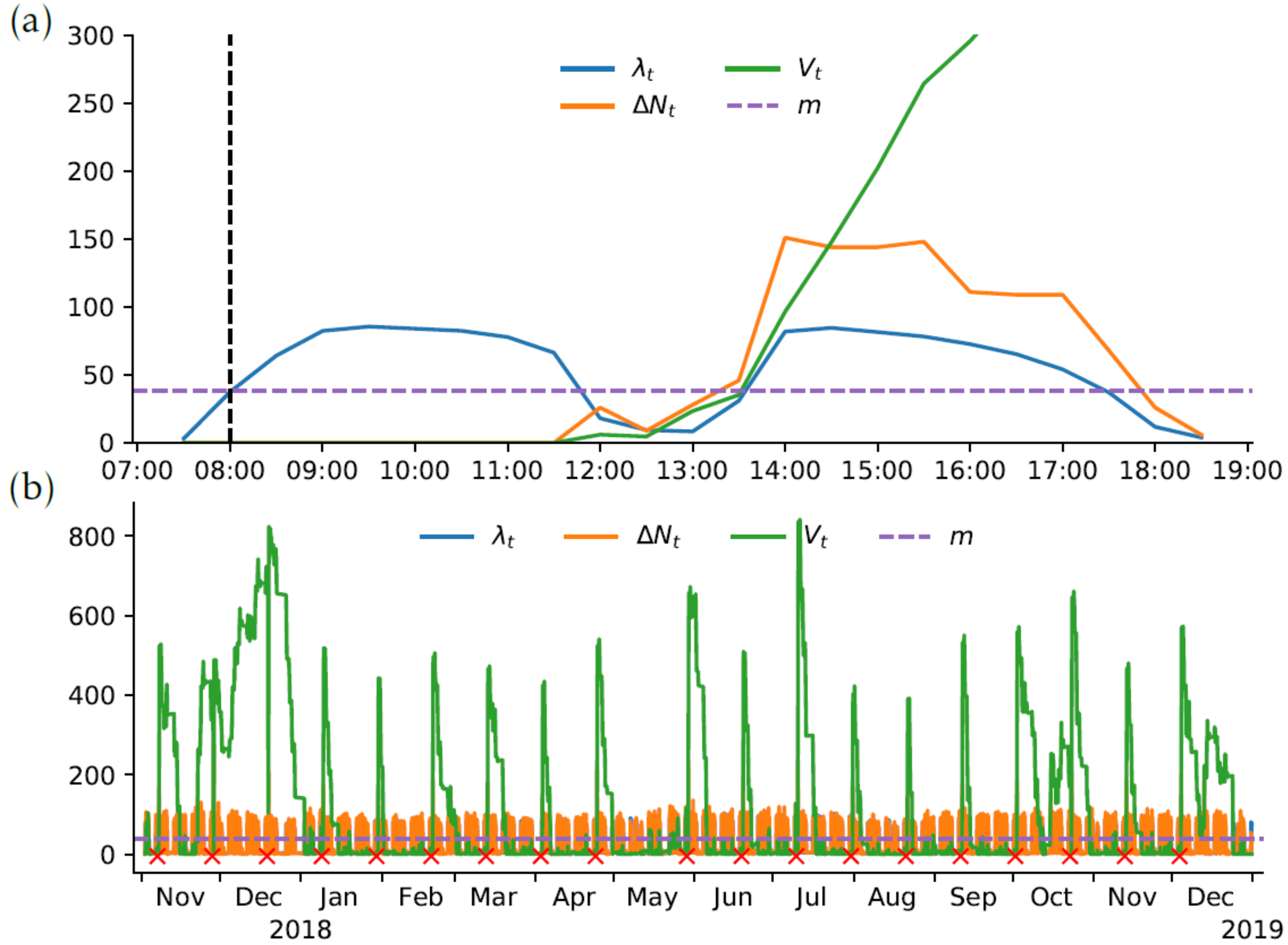



Figure 11: The CUSUM results over (a) 2018-04-03 and (b) the whole test set, where every third Tuesday (marked with a red cross) has had the morning calls shifted into the afternoon.

# What if some calls are postponed? Still works!

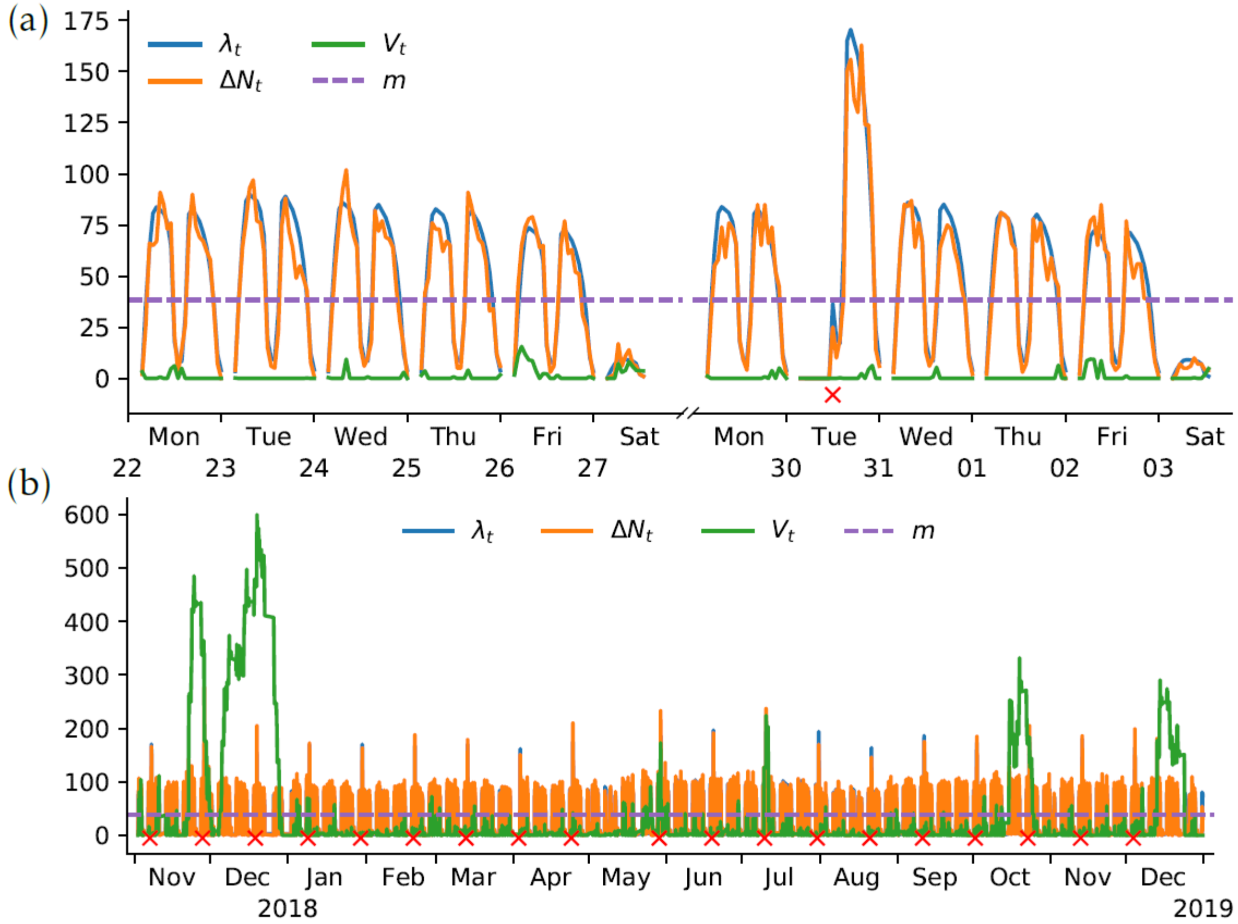



Figure 12: The refitted CUSUM results over (a) a fortnight from 2018-01-22 and (b) the whole test set, where every third Tuesday (marked with a red cross) has had the morning calls shifted into the afternoon.

# Potential next steps with 

- More detailed presentation with Icare team
- Specific detection for some garage / type of car / region
- Multiple cusums
- Combine with predictive modeling

# Wrap-up

- Even if it is not easy to show its optimality,
- Cusum is simple to implement and easy to visualize
- It is faster than anything else (in particular moving window method)
- No miracle!