Mean-variance hedging of unit linked life insurance contracts in a Lévy model

joint work with Frank Bosserhoff and Mitja Stadje

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Overview

motivation

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solution

conclusion

motivation

Life insurance companies face (mainly) the following risk types

- asset risk,
- change in interest rate and
- mortality risk.

Therefore we want to find an optimal asset allocation under these uncertainties.
$\leadsto$ Classical model: Black-Scholes market.

motivation

- Empirical evidence of jumps in stock prices (see e.g. Cont and Tankov):
  Possible reasons: good/ bad news, annual shareholders' meeting
- jumps in interest rate due to changing credibility (ratings) or changes of base rates (ECB)
- We additionally consider mortality risk using a force of mortality with jumps (see Luciano and Vigna 2008 or Cairns et al. 2008).
  Possible reasons: pandemics, wars, medical breakthroughs

$\rightsquigarrow$ Lévy market more realistic

motivation

Looking at a market with jumps: **no** complete financial market.
Problem: Hedging not possible any more.
$\rightsquigarrow$ Other performance criteria like *mean-variance hedging*, expected utility theory, etc.

motivation

We consider a *mean-variance problem*

- classical optimization problem (Markowitz 1952, Merton 1972)
- intuitive understanding of variance as deviation measure
- preference free, no utility function needed
- mathematically tractable (due to L^2 and L^1 theory)

motivation

Classical mean-variance problem for a (discounted) portfolio process $\tilde{P}^u(t)$ (at time $t \in [0, T]$) using an 'admissible' strategy $u \in \mathcal{U}$, depending on a general (discounted) Markov process $\tilde{Z}(t)$:
This has objective function

$$\begin{aligned} J(t, p, z, u) :=& \mathbb{E}[\tilde{P}^u_T | \tilde{P}(t) = p, \tilde{Z}(t) = z] \\ &- \frac{\gamma}{2} \mathrm{Var}(\tilde{P}^u_T | \tilde{P}(t) = p, \tilde{Z}(t) = z) \end{aligned}$$

for some $\gamma > 0$ (non-linearity of variance: MV problem time inconsistent).

motivation

In a *time consistent* problem the Bellman optimality principle holds (see Björk and Murgoci, 2010).

Bellman optimality principle

If a control law is optimal on the full time interval $[0, T]$, then is also optimal on any subinterval $[t, T]$.

In a *time inconsistent* problem this does not hold true, i.e., for some fixed initial point (t, p, z) the control law $\hat{u}$ which maximizes $J(t, p, z, u)$ at that time, is no longer optimal for some later point $(s, \tilde{P}(s), \tilde{Z}(s))$ for the functional $J(s, \tilde{P}(s), \tilde{Z}(s), u)$ (see also Karnan et al., 2017)

motivation

Björk and Murgoci (2010) suggest a reformulation within a game-theoretic framework looking for a Nash subgame perfect equilibrium:

Björk and Murgoci (2010)

They showed that for any time-inconsistent problem there is a time-consistent problem, which is equivalent in some sense.

$\rightsquigarrow$ extended HJB system

In (Markovian) Brownian process setting, see Lindensjö, 2017.

equilibrium control

A trading strategy $u^\star \in \mathcal{U}$ is an *equilibrium control* for the dynamic optimization problem if

$$\liminf_{h\to 0} \frac{J(t,p,z,u^\star) - J(t,p,z,u_h)}{h} \geq 0$$

for any $t \in [0,T]$ and trading strategy $u_h \in \mathcal{U}$ satisfying, for any $u \in \mathcal{U}$,

$$u_h(\tau) = u_\tau \mathbb{1}_{[t,t+h)}(\tau) + u^\star_\tau \mathbb{1}_{[t+h,T]}(\tau), \quad \tau \in [t,T].$$

The *equilibrium value function* is defined by

$$V(t,p,z) := J(t,p,z,u^\star).$$

feedback type

An equilibrium control $u^\star$ is of *feedback type* if, for some feedback function $u_\star : [0, T] \times \mathbb{R}_+ \times \mathbb{R}^5 \to \mathbb{R}$ such that the stochastic differential equation of $\tilde{P}(t)$ and $\tilde{Z}(t)$ has a unique solution $\tilde{P}^\star(t)$ and $\tilde{Z}^\star(t)$, respectively, we have

$$u^\star(t) = u_\star(t, \tilde{P}^\star(t-), \tilde{Z}^\star(t-))$$

with $\tilde{P}^\star(0-) = \tilde{P}^\star(0)$ and $\tilde{Z}^\star(0-) = \tilde{Z}^\star(0)$.

market setting

- $T > 0$ maturity
- $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed probability space, endowed with filtration $(\mathcal{F}_t)_{t\in[0,T]}$
- Let $X = (X_t)_{t\in[0,T]}$, $\bar{X}$ and $\hat{X}$ be Lévy processes, the drivers of asset, mortality and interest rate risk, respectively. ($\rightsquigarrow$ possibly dependent)
- We consider a general stochastic interest rate $r(t)$,
- a stochastic discount factor $D(t) = \exp\left(\int_0^t r(s)ds\right)$,
- two risky assets $S(t)$ and $Y(t)$
- and a zero-coupon-bond $B(t, r(t))$, which pays 1 unit at maturity T.
- The Portfolio process is self-financing.

market setting

To our Lévy process X, we denote the associated Brownian motion by W and $J_X(dt, dx)$ denotes the Poisson random measure of the (independent) Lévy jump process (by Lévy-Itô-decomposition). For the Lévy measure $\vartheta_X(dx)$ we have the assumption $\int_{\mathbb{R}\setminus\{0\}} |x|^2 \vartheta_X(dx) < \infty$. So we can write

$$\tilde{J}_X(dt, dx) := J_X(dt, dx) - \vartheta_X(dx)dt$$

for the *compensated* Poisson random measure.
Analogous definition of $\hat{W}$, $J_{\hat{X}}(dt, d\hat{x})$, $\vartheta_{\hat{X}}(d\hat{x})$, $\tilde{J}_{\hat{X}}(dt, d\hat{x})$ and $\bar{W}$, $J_{\bar{X}}(dt, d\bar{x})$, $\vartheta_{\bar{X}}(d\bar{x})$, $\tilde{J}_{\bar{X}}(dt, d\bar{x})$.

market setting

The Lévy process $\hat{X}$ is the driver of the *stochastic interest rate* modelled by the dynamics

$$
\begin{aligned}
dr(t) =& \mu_r(t, r(t))dt + \sigma_r(t, r(t))d\hat{W}(t) \\
&+ \int_{\mathbb{R}\setminus\{0\}} \eta_r(t, r(t), \hat{x})\tilde{J}_{\hat{X}}(dt, d\hat{x}).
\end{aligned}
$$

Therefore, we consider the *discounted problem*, i.e., we take the discount factor $D(t) = \exp(\int_0^t r(s)ds)$ as the numeraire to all relevant quantities (c.f. Basak and Chabakauri, 2010).
The strategy $\tilde{u}(t) = (\tilde{u}_B(t), \tilde{u}_S(t), \tilde{u}_Y(t))_{t\in[0,T]}$ denotes the discounted total position in the zero-coupon Bond B, stock S and mortality bond Y.

market setting

We consider a zero-coupon-bond $(B(t, r(t)))_{t\in[0,T]}$, which pays 1 unit at maturity T.
We assume that $(r(t), B(t, r(t))))_{t\in[0,T]}$ to be a Markov process. Further we assume that the discounted zero-coupon bond $\tilde{B}(t, r(t)) = \frac{B(t,r(t))}{D(t)}$ has the following representation

$$\begin{aligned}\frac{d\tilde{B}(t, r(t))}{\tilde{B}(t-, r(t-)))} =& (\mu_B(t, r(t), B(t, r(t))) - r(t))\, dt \\ &+ \sigma_B(t, r(t), B(t, r(t)))d\hat{W}(t) \\ &+ \int_{\mathbb{R}\setminus\{0\}} \eta_B(t, r(t), B(t, r(t)), \hat{x})\tilde{J}_{\hat{X}}(dt, d\hat{x}).\end{aligned}$$

market setting

The price of the discounted risky asset $\tilde{S} = (\tilde{S}(t))_{0 \le t \le T}$ is given by the discounted exponential of the Lévy process $X(t)$, namely $\tilde{S}(t) = \frac{e^{X(t)}}{D(t)}$ (only valid under Assumption $\int_{|x| \ge 1} e^x \vartheta_X(dx) < \infty$). This asset has dynamics

$$\frac{d\tilde{S}(t)}{\tilde{S}(t-)} = (\mu_S - r(t))dt + \sigma dW(t) + \int_{\mathbb{R} \setminus \{0\}} \eta_S(x) \tilde{J}_X(dt, dx)$$

with $\mu_S \in \mathbb{R}$, $\sigma > 0$, $\eta_{\tilde{S}}(x) = e^x - 1$.

Further we assume that $(r(t), \tilde{S}(t, r(t)))_{t \in [0,T]}$ is a Markov process.

market setting

We model a longevity bond, bought at time $t_1 < T$ with terminal payoff $\frac{l(T)}{l(t_1)}$ (quotient of survivors). It has dollar value at intermediate time t_2 ($t_1 < t_2 < T$)

$$Y(t_2) = e^{-\int_{t_1}^{t_2} \lambda(s)ds} \, \mathbb{E}_{\mathbb{Q}} \left[\exp\left(-\int_{t_2}^{T} (r(s) + \lambda(s)) \, ds \right) \middle| \mathcal{F}_{t_2} \right]$$

with force of mortality given by

$$\begin{aligned} d\lambda(t) =& \mu_\lambda(t, \lambda(t))dt + \sigma_\lambda(t, \lambda(t)) d\bar{W}(t) \\ & + \int_{\mathbb{R}\setminus\{0\}} \eta_\lambda(t, \lambda(t), \bar{x}) \tilde{J}_{\bar{X}}(dt, d\bar{x}). \end{aligned}$$

We assume $\lambda(t) \geq 0$ $\mathbb{P}$-a.s. (compare discussion in Luciano and Vigna 2008).

market setting

Assuming that $(r(t), Y(t))_{t\in[0,T]}$ is a Markov process, one can show the representation for the discounted longevity asset $\tilde{Y}(t) = \frac{Y(t)}{D(t)}$ as

$$\frac{d\tilde{Y}(t)}{\tilde{Y}(t-)} = (\mu_Y(t, r(t), \lambda(t), Y(t)) - r(t))\, dt + \sigma_Y(t, r(t), \lambda(t), Y(t))^{\mathsf{T}} d\vec{W}(t)$$
$$+ \int_{\mathbb{R}^3\setminus\{0,0,0\}} \eta_{\tilde{Y}}(t, r(t), \lambda(t), Y(t), \hat{x}, x, \bar{x}) \tilde{J}_{\hat{X},X,\bar{X}}(dt, d\hat{x}, dx, d\bar{x})$$

with μ_Y, $\eta_Y \in \mathbb{R}$ and $\sigma_Y \in \mathbb{R}^3$.
Here $\vec{W}(t) = (\hat{W}(t), W(t), \bar{W}(t))$ denotes the vector of the Brownian motions of the underlying Lévy processes.

market setting

We define the 5-dimensional process $(\tilde{Z}(t))_{t\in[0,T]} := (r(t), \tilde{B}(t), \tilde{S}(t), \lambda(t), \tilde{Y}(t))_{t\in[0,T]}$, which is a Markov process.
Further the discounted portfolio equation is given by (self-financing)

$$d\tilde{P}(t) = \frac{\tilde{u}_B(t-)}{\tilde{B}(t-)} d\tilde{B}(t) + \frac{\tilde{u}_S(t-)}{\tilde{S}(t-)} d\tilde{S}(t) + \frac{\tilde{u}_Y(t-)}{\tilde{Y}(t-)} d\tilde{Y}(t).$$

We denote by $\Delta\tilde{Z}(t, \hat{x}, x, \bar{x})$ and $\Delta\tilde{P}(t, \hat{x}, x, \bar{x})$ the jump part of the processes, respectively.

Definition

Let $\mathcal{L}^u : C^{2,2}(\mathbb{R}_+ \times \mathbb{R}^5, \mathbb{R}) \to C^{0,0}(\mathbb{R}_+ \times \mathbb{R}^5, \mathbb{R})$, $f \mapsto \mathcal{L}^u f$ be the *infinitesimal generator* of $(\tilde{P}, \tilde{Z})$ is given by

$$\begin{aligned}&\mathcal{L}^u_{t,p,z}(f(\tilde{P}(t), \tilde{Z}(t))) \\ &:= \lim_{\varepsilon \searrow 0} \frac{\mathbb{E}_{t,p,z}[f(\tilde{P}(t+\varepsilon), \tilde{Z}(t+\varepsilon)) - f(\tilde{P}(t), \tilde{Z}(t))]}{\varepsilon}.\end{aligned}$$

Then we can define the *infinitesimal generator of the graph* by $\mathcal{A}^u : C^{0,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R}^5, \mathbb{R}) \to C^{0,0}(\mathbb{R}_+ \times \mathbb{R}^5, \mathbb{R})$, $f \mapsto \mathcal{A}^u f$ via

$$\mathcal{A}^u f(t,p,z) := \dot{f}(t,p,z) + \mathcal{L}^u_{t,p,z} f(p,z),$$

where $\dot{f}(t,p,z) := \frac{\partial f}{\partial t}(t,p,z)$.

For a function $f \in C^{2,2}(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$, the infinitesimal generator is given by (for $t \in [0, T]$, $(p, z) \in \mathbb{R}^{1+5}$ and $\tilde{u} \in \mathcal{U}$)

$$
\begin{aligned}
\mathcal{L}^{u}_{t,p,z}(f(p,z)) = & f_p(p,z)\mu^{\tilde{u}}_{\tilde{P}}(t,z) + \nabla_z f(p,z)^{\mathsf{T}} \mu_{\tilde{Z}}(t,z) + \frac{1}{2} f_{pp}(p,z)\sigma^{\tilde{u}}_{\tilde{P}}(t,z)^{\mathsf{T}} \Sigma_W(t) \sigma^{\tilde{u}}_{\tilde{P}}(t,z) \\
& + \sum_{i=1}^{5} \nabla_z f_p(p,z)_i \sigma_{\tilde{Z}}(t,z)_{i,:} \Sigma_W(t) \sigma^{\tilde{u}}_{\tilde{P}}(t,z) \\
& + \frac{1}{2} \sum_{i,j=1}^{5} H_{\tilde{Z}} f(p,z)_{i,j} \sigma_{\tilde{Z}}(t,z)_{i,:} \Sigma_W(t) \sigma_{\tilde{Z}}(t,z)_{j,:} \\
& + \int_{\mathbb{R}^3 \setminus \{0,0,0\}} \Big(f(p + \Delta\tilde{P}^u(t,\hat{x},x,\bar{x}), z + \Delta\tilde{Z}(t,\hat{x},x,\bar{x})) - f(p,z) \\
& - (f_p(p,z), \nabla_z f(p,z))^{\mathsf{T}} \left(\Delta\tilde{P}^u(t,\hat{x},x,\bar{x}), \Delta\tilde{Z}(t,\hat{x},x,\bar{x}) \right) \Big) \vartheta_{\hat{X},X,\bar{X}}(d\hat{x}, dx, d\bar{x}).
\end{aligned}
$$

solution

Then the mean-variance problem can be reformulated:

$$
\begin{aligned}
\dot{V}(t,p,z) + \sup_{u \in U} \left\{ \mathcal{L}^u_{t,p,z} V(p,z) + \mathcal{G}^u g(t,p,z) \right\} &= 0 \\
\mathcal{A}^{u^\star} f(t,p,z) &= 0 \\
\mathcal{A}^{u^\star} g(t,p,z) &= 0 \\
V(T,p,z) &= p \\
g(T,p,z) &= p \\
f(T,p,z) &= F(p),
\end{aligned}
$$

where we note that only the first three equations are substantial constraints. The lower remaining equations are directly fulfilled in the mean-variance setting.

solution

Proposition 1

The equilibrium value function $V(t, p, z)$ and the expected terminal payoff $g(t, p, z)$ are given by

$$V(t, p, z) = p + B(t, z),$$
$$g(t, p, z) = p + b(t, z)$$

with $b(t, z)$ and $B(t, z)$ given by conditional expectations.

Proposition 2

The optimal amount of money $\tilde{u}^{\star}(t)$ invested can be calculated explicitly (see next slide).

solution

$$\tilde{u}_B(t-) = \frac{A(t)\left(\mu_B(t,r(t)) - r(t)\right) - \gamma a(t)\sum_{i=1}^{5} \nabla_z b(t,z)_i \left(\frac{\partial}{\partial \tilde{u}_B}\sigma_{\tilde{Z}}(t,\tilde{Z}(t))\Sigma_W(t)\sigma_{\tilde{P}}^{\tilde{u}}(t,\tilde{Z}(t))\right)_i}{\gamma a(t)^2\left(\sigma_B(t,r(t),B(t))^2\Sigma_W(t)_{1,1} + \int_{\mathbb{R}\setminus\{0\}} \eta_B(t,r(t),B(t),\hat{x})^2\vartheta_{\hat{X}}(d\hat{x})\right)}$$
$$- \frac{\int_{\mathbb{R}^3\setminus\{0,0,0\}}\left(\left(b(t,z+\Delta\tilde{Z}(t,\hat{x},x,\bar{x})) - b(t,z)\right)\eta_B(t,r(t),B(t),\hat{x})\right)\vartheta_{\hat{X},X,\bar{X}}(d\hat{x},dx,d\bar{x})}{a(t)\left(\sigma_B(t,r(t),B(t))^2\Sigma_W(t)_{1,1} + \int_{\mathbb{R}\setminus\{0\}} \eta_B(t,r(t),B(t),\hat{x})^2\vartheta_{\hat{X}}(d\hat{x})\right)}$$
$$- \tilde{u}_S(t-)\frac{\sigma_B(t,r(t),B(t))\sigma\Sigma_W(t)_{1,2} + \int_{\mathbb{R}^3\setminus\{0,0,0\}}\left(\eta_S(x)\eta_B(t,r(t),\hat{x})\right)\vartheta_{\hat{X},X,\bar{X}}(d\hat{x},dx,d\bar{x})}{\sigma_B(t,r(t),B(t))^2\Sigma_W(t)_{1,1} + \int_{\mathbb{R}\setminus\{0\}} \eta_B(t,r(t),B(t),\hat{x})^2\vartheta_{\hat{X}}(d\hat{x})}$$
$$- \tilde{u}_Y(t-)\frac{\sigma_B(t)\sigma_Y(t)^{\mathsf{T}}\Sigma_W(t)_{1,:} + \int_{\mathbb{R}^3\setminus\{0,0,0\}}\left(\eta_Y(t,\hat{x},x,\bar{x})\eta_B(t,r(t),\hat{x})\right)\vartheta_{\hat{X},X,\bar{X}}(d\hat{x},dx,d\bar{x})}{\sigma_B(t,r(t),B(t))^2\Sigma_W(t)_{1,1} + \int_{\mathbb{R}\setminus\{0\}} \eta_B(t,r(t),B(t),\hat{x})^2\vartheta_{\hat{X}}(d\hat{x})}$$

and similar expressions for $\tilde{u}_S(t-)$ and $\tilde{u}_Y(t-)$.
$\rightsquigarrow$ system of linear equations

Theorem: Sufficiency of the HJB system

Assume that $V \in C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R}^5, \mathbb{R})$, $f, g \in C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}^5, \mathbb{R})$ solve the extended HJB system and that the supremum in the first line is attained for every $(t,p,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^5$. Then there exists an equilibrium law $u^\star$ and it is given by the optimal u in the first equation in the HJB. Furthermore, V is the corresponding value function and f, g have the stated form.

Theorem: Necessity of the HJB system

Assume there exists an equilibrium law $u^\star$ and that V is the corresponding value function. Then V satisfies the extended HJB system and $u^\star$ attains the supremum in the first equation of the HJB system.

conclusion

- We extend the (Markovian) Itô process setting to general Lévy processes with additional Mortality bond and stochastic interest rate.
- In (time-inconsistent) mean-variance setting, we derive an (time-consistent) extended HJB equation.
- We prove necessity and sufficiency of the HJB equation approach.
- We calculate the solutions explicitly.

Thank you for your attention!

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